Modelling CO_2 permit prices

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Contents

List of Abbreviations

AAU	Assigned Amount Unit
BAU	business as usual
bn	billion (1,000,000,000)
c.d.f.	cumulative distribution function
CCS	Carbon Capture and Storage
CDM	Clean Development Mechanism
CER	Certificate of Emission Reduction
cf.	confer
CH_4	methane
CITL	Community Independent Transaction Log
CO_2	carbon dioxide
CO_2-e	carbon dioxide equivalent
COP	Convention of the Parties (Signatories of the UNFCCC)
CPR	Commitment Period Reserve
CPRS	Carbon Pollution Reduction Scheme (Australia)
CT	Chesney and Taschini (2008)
Def.	Definition
EC	European Commission
ECX	European Climate Exchange
EEC	Expected Enforment Costs
EIT	Economies in Transition
erf	Error Function
erfc	Complementary Error Function
ERU	Emission Reduction Unit
ETS	Emissions Trading Scheme
EU	European Union

Abbreviation	**Full name**
EU ETS	European Union Emission Trading Scheme
EU-15	Member states of the EU in 1995
EU-27	Member states of the EU in 2010 (last enlargement in 2007)
EUA	European Union Allowance
EUA1	EUA for Phase I (2005-2007)
EUA2	EUA for Phase II (2008-2012)
EUA3	EUA for Phase III (2013-2020)
EUA-Dec07	Futures contract on EUA that matures in December 2007
EUA-Dec12	Futures contract on EUA that matures in December 2012
GARCH	Generalized Autoregressive Conditional Heteroscedasticity
GBM	geometric Brownian motion
GDP	Gross Domestic Product
GHG	greenhouse gas
GWP	global warming potential
HFC	hydrofluorocarbon
i.e.	id est
IG	Inverse Gamma/ Reciprocal Gamma
IPCC	Intergovernmental Panel on Climate Change
ITL	International Transaction Log
JI	Joint Implementation
KP	Kyoto Protocol
KS	Kolmogorov-Smirnov
L.	Lemma
lCER	long-term CER
LDC	Least Developed Countries
L'Hop.	L'Hôpital's rule
LULUCF	Land Use, Land Use Change and Forestry
m	million (1,000,000)
N_2O	nitrous dioxide
NAP	National Allocation Plan
NIG	Normal Inverse Gaussian
NZ ETS	New Zealand's Emissions Trading Scheme
OECD	Organisation for Economic Co-operation and Development
p.d.f.	probability density function

Abbreviation	**Full name**
pCER	primary CER
PDE	partial differential equation
PFC	perfluorocarbon
PhD	Doctor of Philosophy
Q-Q	Quantile-Quantile
RGGI	Regional Greenhouse Gas Initiative
RMU	Removal Unit
sCER	secondary CER
SDE	stochastic differential equation
SF_6	sulphur hexafluoride
tCER	temporary CER
TEHG	Treibhaus-Emissionshandelsgesetz
Th.	theorem
UK	United Kingdom
UNFCCC	United Nations Framework Convention on Climate Change
UN-OHRLLS	UN Office of the High Representative for the Least Developed Countries, Landlocked Developing Countries and Small Island Developing States
US	United States of America
w.l.o.g.	without loss of generality
w.r.t.	with respect to
ZuG	Zuteilungsgesetz

List of Symbols

dW_t	increment of a standard Brownian motion
$e^x = \exp(x)$	exponential function
$\mathbb{E}[X \mid \mathcal{F}_t]$	conditional expectation of the random variable X
$\mathcal{F}_t$	filtration at time t
$\Phi(\cdot)$	standard normal distribution function
$\Phi^{-1}(\cdot)$	quantile function of the standard normal distribution
$X \sim \Gamma(\alpha, \beta)$	X is a Gamma distributed random variable with shape parameter α and scale parameter β
$X \sim IG(\alpha, \beta)$	X is reciprocal gamma distributed with parameters α and β
$\ln(x)$	natural logarithm
$X \sim \log N(\mu, \sigma^2)$	X is distributed log-normally with parameters μ and σ^2
$X \sim N(\mu, \sigma^2)$	X is distributed normally with parameters μ and σ^2
$X \sim N(0, 1)$	X is standard normally distributed
$\mathbb{P}$	historical measure
$\mathbb{Q}$	risk neutral measure
$\mathbb{R}$	ordinary real line $(-\infty, \infty)$
$\mathbb{R}^+$	positive real line $(0, \infty)$
W_t	standard Brownian motion
$\mathbf{1}_{\{x \in A\}}$	indicator function: takes the value 1 if $x \in A$ and 0 if $x \notin A$
$(\cdot)^+$	maximum of the argument and zero

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Chapter 1

Introduction

1.1 Background

Global warming has gradually become a serious threat to our world. Voices of climatologists making anthropogenic greenhouse gas emissions responsible for the rise in global temperatures cannot be ignored any longer. If the world's leaders are willing to change climate policy there is a chance of avoiding the worst consequences, however.
Without a different policy regarding climate change mankind will be confronted with dire consequences. For instance, the report of Stern (2007) predicts that rising sea levels and more extreme weather such as storms, floods and droughts will displace millions of people. Europe might face dramatic changes in the weather if the Gulf Stream will change its course due to melting icebergs in the Arctic. Melting glaciers in many parts of the world might lead to a shortage of drinking water and growing deserts to diminishing areas of cultivable land. Climate scientists argue that the worst consequences of global warming can be avoided by reducing anthropogenic greenhouse gas emissions. Cost-benefit analyses of Stern (2007) and McKinsey & Company (2009) show that the measures required to prevent global temperatures from rising too high will be affordable. Politicians can achieve a reduction in greenhouse gas emissions by implementing suitable instruments.

In other words, policymakers have to control pollutions. Desired characteristics of the policy response to the pollution control problem are

1. Effectiveness
   Policy instruments should ensure that the targeted pollution level is achieved.

2. Manageable control effort
   The control authority should be able to measure emissions or to estimate them correctly at reasonable costs.

3. Minimal costs from a macroeconomic perspective
   The policy instrument ensures that emitters with the lowest emission reduction costs per unit of emission (marginal abatement costs) reduce their emissions at first.

In principle, possible policy responses to the pollution control problem include

1. Emission standards ("Command-and-Control")
   Legal limits on the amount of pollutants an individual source is allowed to emit.

2. Taxes

   (a) Emission charges
       Polluter has to pay a fee for each unit of pollutant emitted.

   (b) Product charges
       Control authority taxes the commodity that is responsible for the pollution instead of the pollutant.

3. Subsidy
   Control authority incentivizes polluters to emit less by offering them a subsidy per unit of reduced emission.

4. Emissions trading
   Generally speaking, emissions trading consists of three major steps:

   (a) Allocation: Control authority allocates emission allowances to the emitters. The total number of allowances, also known as cap, is chosen by the control authority according to its reduction target. An emission allowance gives the holder the right to emit one unit during a pre-specified period of time, also called compliance period. The most popular allocation methods are grandfathering and auctioning. Grandfathering means that allowances are allocated for free and that the number of allowances is related to the emissions of each emitter in the baseline year.

   (b) Trading: The allowances are freely tradeable.

(c) Compliance: At the end of the compliance period emitters have to hand in one allowance per unit of emission. To enforce the cap, a penalty is levied for each unit of pollutant emitted outside the limits of a given compliance period.

The emissions trading scheme with grandfathering as described above is referred to as **ordinary scheme** in the following in order to distinguish it from the so-called hybrid schemes (cf. page 4).

The thesis mainly focuses on emissions trading because with the launch of the European Union Emission Trading Scheme (EU ETS) in 2005, marketable permits have become the cornerstone of the policy of the European Union towards climate change.
Moreover, the introduction of a cap-and-trade system is being discussed in several countries all over the world. The official launch of the Regional Greenhouse Gas Initiative (RGGI) in January 2009 which was signed by 10 north-eastern US States was the beginning of the carbon market era in North America. A plan to introduce a US-wide cap-and-trade scheme has recently been proposed by the new U.S. administration. Canada demonstrated its interest in linking up with the US scheme, abandoning its own plans for developing an efficiency-based system. Schemes in the Pacific area, such as Australia's Carbon Pollution Reduction Scheme (CPRS) and New Zealand's Emissions Trading Scheme (NZ ETS) are in different stages of development. Finally, Japan is timidly considering different options for the development of a market for emission permits.

The work of Coase (1960) on externalities laid the basis for the theory of marketable permits. Coase sees undefined property rights as the root of the problem with externalities such as greenhouse gas emissions. According to his theory a society will reduce externalities to a socially acceptable level by legal and by market mechanisms in the presence of defined property rights such as rights for clean air, clean drinking water, etc. Dales (1968) was the first to propose the introduction of marketable permits.
A few years later Montgomery (1972) developed the first theoretical model for permit prices in a cap-and-trade-system. He formally demonstrated that emissions trading reduces emissions at minimal costs, i.e. emissions trading satisfies the third desired characteristic of a pollution control instrument. The rationale behind this result is that in a cap-and-trade system firms may either reduce their own pollution or purchase emission permits in order to ensure compliance. Firms that can easily reduce emissions will do so, whereas those firms unable to reduce emissions will buy permits.
Due to its construction a cap-and-trade system ensures that the required emission level is not exceeded and thereby effectively ensures the reduction target (first desired charac-

teristic of the policy instrument). For this reason, emissions trading is called a quantity instrument in contrast to price instruments such as a tax or a subsidy that reduce emissions indirectly. When focusing on the largest emitters only, control efforts of cap-and-trade systems are manageable (second characteristic in the list).
This explains the popularity of cap-and-trade systems among policymakers.

A clear understanding of the carbon pricing mechanism is necessary because of the prevalence of cap-and-trade schemes. The EU ETS (European Union Emission Trading Scheme) is by far the largest CO_2 emissions trading system in the world covering about 50% of all CO_2 emissions in the European Union. Permit markets of other CO_2 emissions trading schemes are still relatively illiquid.
Permit prices of the EU ETS showed the following peculiarities:

- Jumpy behaviour
- Spot price converged to zero at the end of the first compliance period.

This price behaviour led to discussions about acceptable price ranges for emission permits. There are proposals suggesting the introduction of specific mechanisms to keep the permit price from rising too high or falling too low. These modified ordinary schemes are called **hybrid schemes**.

1.2 Aim of the thesis

The aim of this PhD thesis is to provide a theoretical explanation of those two permit price characteristics in an ordinary scheme and to check whether the proposed hybrid schemes are able to avoid the two characteristics (jumpy behaviour and convergence of the spot price to zero at the end of the compliance period).
Before performing these analyses it is necessary to investigate deterministic equilibrium models and two classes of models that have been developed recently, namely **stochastic equilibrium models** and **reduced-form models**. Special attention is paid to stochastic equilibrium models as permit price dynamics cannot be captured by deterministic models.

1.3 Contribution of the thesis

1. Investigation of deterministic and stochastic equilibrium models and reduced-form models

   (a) Show how the two concepts of marginal abatement costs and the probability of permit shortage are related to each other (cf. Section 3.4)

   (b) Develop a new stochastic equilibrium model (cf. Section 3.2d)

   (c) Show how stochastic equilibrium models and reduced-form models are related to each other (cf. Section 3.5)

   Table 1.1 (page 6) contains a compilation of permit price models.

2. Analysis of the price dynamics in an ordinary scheme

   (a) Provide a theoretical explanation for the observed jumpy permit price behaviour (cf. Section 4.1)

   (b) Explain why the spot price converged to zero at the end of the first compliance period (cf. Section 3.2 and 4.2)

   (c) Discuss estimation methods for stochastic equilibrium models and reduced-form models (cf. Section 4.3)

3. Analysis of the permit prices in the proposed hybrid schemes (cf. Chapter 5)

   (a) Analyze the relationship between an ordinary scheme and hybrid schemes

   (b) Determine the effectiveness of price bounds in hybrid schemes

   (c) Investigate permit price volatility in hybrid schemes

   (d) Analyze enforcement costs and environmental targets in hybrid schemes

Figure 1.1 (page 7) provides a survey on the structure of the thesis.

Permit price model	**Section**
Deterministic equilibrium model	
Montgomery (1972)	3.1a
Rubin (1996)	3.1b
Kling and Rubin (1997)	3.1c
Cronshaw and Kruse (1996)	3.1d
Stochastic equilibrium model	
Seifert et al. (2008)	3.2a
Carmona et al. (2009b)	3.2b
Chesney and Taschini (2008)	3.2c
Grüll and Kiesel (2009)	3.2d
Reduced-form model	
Carmona et al. (2009a)	3.3a
Grüll and Taschini (2009)	3.3b

Table 1.1: Overview of the permit price models (deterministic and stochastic equilibrium models and reduced-form models).

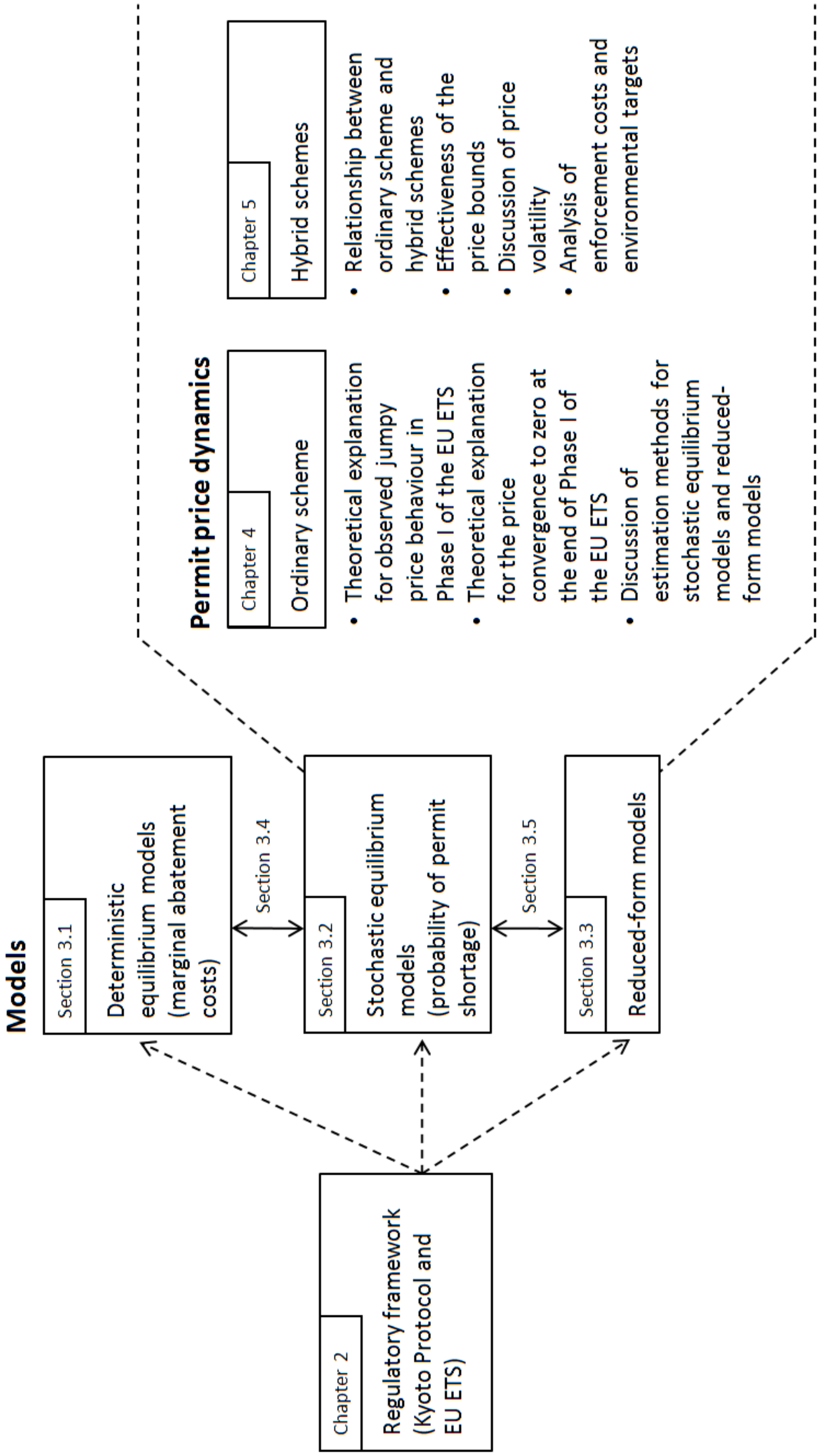

Figure 1.1: Structure of the PhD thesis.

Chapter 2

Regulatory Framework

2.1 Kyoto Protocol

2.1a Introduction

The first international agreement in reaction to increasing worldwide changes in climate was the United Nations Framework Convention on Climate Change (UNFCCC). It came into force on 21 March 1994 after having been ratified by 192 nations. With the ratification of the United Nations Framework Convention on Climate Change (UNFCCC) the world's political leaders admitted that anthropogenic emissions of carbon dioxide and other greenhouse gases do have a significant impact on the climate change. Greenhouse gases (GHG) besides carbon dioxide (CO_2) are methane (CH_4), nitrous oxide (N_2O), sulphur hexafluoride (SF_6), hydrofluorocarbons (HFCs) and perfluorocarbons (PFCs). The United Nations Framework Convention on Climate Change (UNFCCC) encourages developed countries to stabilize their emissions of greenhouse gases (GHG) but there is no legal obligation to do so.

The first international agreement with a binding obligation to reduce the emissions of greenhouse gases is the Kyoto Protocol. It was adopted on 11 December 1997 and came into force on 16 February 2005. Not all the countries that had ratified the United Nations Framework Convention on Climate Change (UNFCCC), also ratified the Kyoto Protocol, however, as this agreement is not non-committal any longer. The United States of America signed it but refused to ratify it in the end. The Kyoto Protocol obliges Annex-I countries to reduce their collective annual greenhouse gas emissions in the period between 2008 and 2012 by 5.2% compared to the emission level of the year 1990. In other words, Annex-I countries on average have an emission reduction target of 5.2% compared to the baseline year 1990. Annex-I countries are Western industrialized nations such as the member states of the European Union, the United States of America, Canada, Japan, Australia and New Zealand. Moreover, countries having emerged from the decaying Soviet bloc belong to the list of Annex-I countries. Those former Communist nations are referred to as Economies in Transition (EIT). The Non-Annex-I countries that have ratified the Kyoto Protocol have not committed themselves to any emission reduction. Nevertheless, the Kyoto Protocol is a joint agreement between Annex-I and Non-Annex-I countries. This has to do with the agreement that Annex-I countries are allowed to achieve parts of their emission reduction commitment in form of emission reduction projects in Non-Annex-I countries. This mechanisms is called Clean Development Mechanism and it is one of the flexible mechanisms of the Kyoto Protocol that are described later on (Section 2.1c). The list of Non-Annex-I countries comprises both, the poorest nations which are referred to by Least Developed Countries (LDC) and emerging economies such as China, India, Brazil and also the wealthy Gulf nations and South Korea[1].

The emissions in the baseline year 1990 determine the Assigned Amount, i.e. the total number of emission allowances that an Annex-I country receives for the first compliance period between 2008 and 2012. One AAU (Assigned Amount Unit) gives the holder the right to emit one ton of CO_2-e during the compliance period. At the end of the compliance period an Annex-I country has to hand in one AAU per unit of emission (one ton of CO_2-e). If an Annex-I country fails to do so it faces the following penalty for each unit of emission that is not covered by an AAU. A non-compliant Annex-I country is not released from the obligation to hand in the lacking permit in the second compliance period (after 2012). Furthermore, it has to hand in 0.3 additional permits in the course of the second compliance period.

The penalty will only affect non-compliant countries if there is a second compliance period. However, political leaders failed to negotiate a Post-Kyoto Protocol so far. Therefore, it is uncertain at the moment whether there will be a second compliance period.

Under the assumption that there will be a penalty, the Kyoto Protocol incentivizes countries to hold an AAU for every unit of emission. This means that Annex-I countries have to either reduce their emissions or to increase the number of emission allowances with the help of the flexible mechanisms of the Kyoto Protocol.

Annex-I countries are obliged to achieve compliance mainly by reducing national emissions. National measures include emission reduction projects in the industry and emission reductions in the private sector, e.g. emissions from transportation could be reduced by

[1]A full list of Annex-I, EIT, Annex-II, Non-Annex-I and LDC countries can be found in Section 2.3a.

levying higher taxes on fuel. Moreover, Annex-I countries are allowed to use a limited amount of permits for compliance that result from domestic LULUCF activities (Land Use, Land Use Change and Forestry) such as afforestation and reforestation. LULUCF activities do not necessarily reduce emissions. The reason why they are eligible for compliance is that sinks such as a newly planted forest do not reduce emissions but capture parts of the carbon dioxide in the atmosphere.
In addition to the national measures, Annex-I countries can increase the number of permits that are functional for compliance[2] by making use of the following three flexible market-based instruments of the Kyoto Protocol:

- Emissions Trading
- Joint Implementation (JI)
- Clean Development Mechanism (CDM)

Emissions trading in the context of the Kyoto Protocol means that Annex-I countries can trade emission permits among each other. In contrast to the other two mechanisms, the permits transferred from one country to another do not necessarily result from an emission reduction project. A country that was generously awarded permits (i.e. countries that have negotiated a favourable baseline) might sell those permits to other countries even though the permits might simply originate from overallocation. Especially, countries of the former Soviet bloc (EIT countries) have profited from the fact that the year 1990 had been agreed on as baseline year. Economies of the EIT countries collapsed around 1990 and as a result of this collapse emissions dropped significantly and have never rebounded since then. Therefore, the Emissions Trading mechanism of the Kyoto Protocol has been criticized for creating "Hot Air" in the EIT countries.
Both Joint Implementation (JI) and the Clean Development Mechanism (CDM) allow Annex-I countries to carry out emission reduction projects in other countries and to use the resulting permits for compliance. Depending on the location of the project, it is either referred to as JI project or CDM project. A JI project is located in an Annex-I country. A CDM project is located in a Non-Annex-I country. The most popular host of CDM projects is China. The basic idea behind those flexible mechanisms is to help Annex-I countries in achieving their reduction target and in promoting technology transfer and foreign investment in developing countries and nations under transition.

[2]Section 2.1c provides a detailed survey on the different permits that are functional for compliance with the Kyoto Protocol.

World leaders discussed a possible Post-Kyoto Protocol at the United Nations Climate Change Conference in Copenhagen between 7 December 2009 and 19 December 2009. The negotiations did not result in an international agreement. Therefore, it is uncertain at the moment whether there will be an international agreement with binding reduction targets for the years after 2012.

The UNFCCC Secretariat is responsible for the implementation of the Kyoto Protocol in practice. The UNFCCC Secretariat is part of the United Nations. Its headquarter is located in Bonn and its 200 employees have the following tasks:

- Prepare the annual Convention of the Parties (COP).
  COP-15 took place in Copenhagen in December 2009 and is better known as Copenhagen Conference. The next Convention will take place in Mexico at the end of 2010.
- Monitor emissions (collection and analysis of GHG emissions data)
- Administer the International Transaction Log (ITL).
  Similar to a bank, the ITL documents the number of permits that the different Annex-I countries hold in their national registries.

2.1b Emissions of Greenhouse Gases (GHG)

Definition of GHG emissions

The emissions of the different greenhouse gases (GHG) are converted into the unit of so-called carbon dioxide equivalent (denoted by CO_2-e) because each greenhouse gas (GHG) has a different effect on global warming. The carbon dioxide equivalent of a specific gas describes the amount of CO_2 that would have the same global warming potential over 100 years (GWP) as one unit of the gas. Table 2.1 summarizes the different carbon dioxide equivalents in the report of Forster et al. (2007) which is part of the fourth assessment report of the Intergovernmental Panel on Climate Change (IPCC). The IPCC is the leading scientific and governmental body that assesses climate change and its potential socio-economic consequences. Al Gore and the IPCC were awarded the Nobel Peace Prize in 2007.

Greenhouse gases	Carbon dioxide equivalent of one unit
Carbon dioxide	1
Methane	21
Nitrous oxide	310
Hydrofluorocarbons	140 - 11,700
Perfluorocarbons	6,500 - 9,200
Sulphur hexafluoride	23,900

Table 2.1: Global warming potential of different greenhouse gases.
Source: Forster et al. (2007).

Annex A emissions include the emissions of the following sectors: energy industry, manufacturing and construction, transport, industrial processes, solvents and other product use, agriculture and waste[3]. It is important to note that emissions associated with land use, land use change and forestry (LULUCF) do not belong to the group of Annex A emissions. LULUCF activities include afforestation, reforestation and deforestation.

Baseline and reduction target

The Kyoto Protocol obliges Annex-I countries to reduce their collective annual Annex A emissions in the first compliance period (2008-2012) by 5.2% compared to the emission level of the year 1990. Table 2.2 provides an overview of the individual reduction targets. Both the choice of the baseline year and the reduction target are the result of political negotiations during the Kyoto Conference in 1997. It is remarkable that the baseline year 1990 coincides with the fall of the former Soviet bloc. Some of those former Communist nations (called EIT countries) agreed on limiting their greenhouse gas (GHG) emissions. They are the only economies in transition with binding targets. When the Kyoto Protocol was negotiated it was already obvious that the GHG emissions of EIT countries were approximately 40% below the level of 1990. This shows that the commitment of the EIT countries is not contributing to a real reduction of GHG emissions. Some EIT countries even succeeded in achieving a more favourable baseline. The following countries have a baseline prior to 1990: Bulgaria (1988), Hungary (the average of annual emissions in the years 1985-1987), Poland (1988), Romania (1989) and Slovenia (1986).

[3] A detailed list of Annex A emissions can be found in UNFCCC (2008).

Country	**Reduction target in 2008-2012 compared to 1990 level**
EU-15, Bulgaria, Czech Republic, Estonia, Latvia, Liechtenstein, Lithuania, Monaco, Romania, Slovakia, Slovenia, Switzerland	-8.0%
US[4]	-7.0%
Canada, Hungary, Japan, Poland	-6.0%
Average of all Annex-I countries	-5.2%
Croatia	-5.0%
New Zealand, Russian Federation, Ukraine	0
Norway	+1.0%
Australia	+8.0%
Iceland	+10.0%

Table 2.2: Emission targets of countries included in Annex B to the Kyoto Protocol. Source: UNFCCC (2008).

The reduction target of the European Union is of special interest as the European Union is a pioneer in the field of environmental economics. The countries of the European Union committed themselves to an ambitious reduction target of the Kyoto Protocol. In 2005 the European Union launched the world's largest CO_2 emission trading scheme in order to achieve its reduction targets. Covering approximately 40% of the GHG emissions in the European Union, emissions trading has become a key instrument of environmental politics in Europe. In 1998 the common target of the member states of the European Union has been split up into individual targets for each member state according to the Burden Sharing Agreement. When ratifying the Kyoto Protocol on 31 May 2002, the European Union reaffirmed the validity of the Burden Sharing Agreement. Table 2.3 lists the reduction targets of the EU-15 countries.

[4]The United States of America have not ratified the Kyoto Protocol.

Country	**Reduction target in 2008-2012 compared to 1990 level**
Luxembourg	-28.0%
Denmark, Germany	-21.0%
Austria	-13.0%
United Kingdom	-12.5%
EU-15	-8.0%
Belgium	- 7.5%
Italy	-6.5%
Netherlands	-6.0%
Finland, France	0%
Sweden	+4.0%
Ireland	+13.0%
Spain	+15.0%
Greece	+25.0%
Portugal	+27.0%

Table 2.3: GHG reduction targets of EU-15 countries under the European Burden Sharing Agreement.
Source: Carbontrust (2009).

National Systems and GHG data

The following subsection provides a short survey on the reporting requirements of Annex-I countries and describes the development of GHG emissions in the last 20 years.

In order to record the level of GHG emissions, each Annex-I country must set up a national system and report its GHG data to the UNFCCC Secretariat. The initial report contains information on the GHG data of the baseline year 1990. Reports on the emissions of the years up to 2007 are on a voluntary basis. However, most Annex-I countries have compiled them. Reporting the GHG emissions of the years 2008-2012 is compulsory and the deadlines for the first and the last annual report are April 2010 and April 2014, respectively. After having evaluated all the reports the UNFCCC will publish the so-called true-up report. Due to the relatively complex reporting process the compliance date (31 March 2015) is more than two years after the end of the first commitment period of the Kyoto Protocol (2008 – 2012).

The quality of the GHG data on the emissions of Annex-I countries is relatively good (cf. Figure 2.1) because data has to be reported to the UNFCCC annually. Annex-I countries slightly reduced their emissions by roughly 4% between 1990 and 2007 and it looks as if Annex-I countries could manage to meet their overall reduction obligation of -5.2%. If GHG emissions are broken down into emissions of EIT Parties (with a reduction of about 40%) and into emissions of the remaining Annex-I countries (with an increase of about 10%) it can be shown that the overall reduction is clearly related to the collapse of the Soviet Union.

The effect of LULUCF (emissions/removals from land use, land-use change and forestry) is minor in the context of the Annex-I group. For some countries, however, the change in GHG levels is heavily affected by LULUCF. The most obvious reduction in GHG levels due to LULUCF effects can be observed for the Baltic nations, whereas Australia's GHG emissions are significantly increased when taking LULUCF into account.

Annex-I Parties slightly reduced their GHG emissions excluding LULUCF (also known as Annex A emissions) between 1990 and 2007 from 18.85 bn tons of CO_2-e to 18.11 bn tons of CO_2-e . Carbon dioxide has by far the largest share in total GHG emissions of Annex-I countries (more than 80%). Moreover, the CO_2 emissions nearly remained the same between 1990 and 2007. The overall reduction in GHG emissions is caused by reductions in CH_4 and N_2O emissions of more than 20%[5].

[5]See UNFCCC (2009).

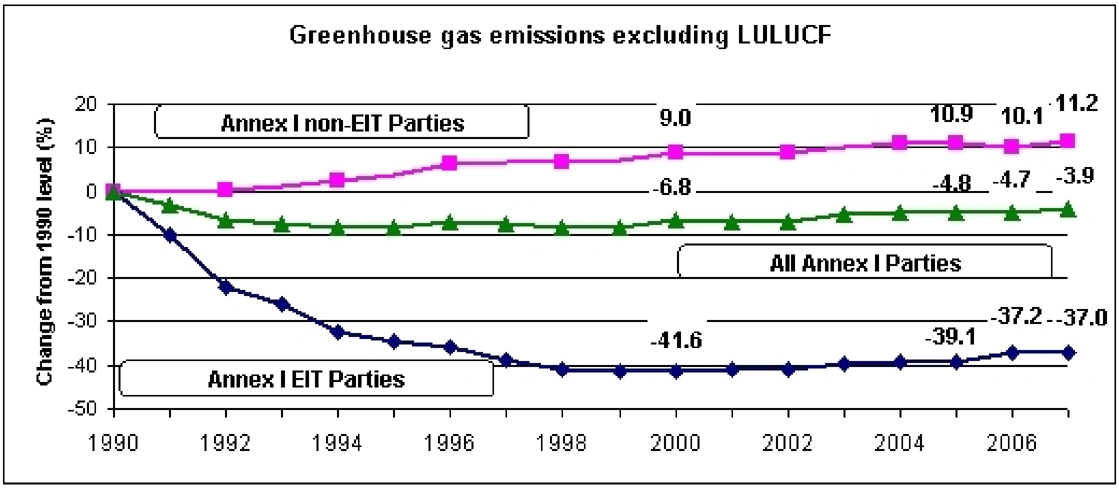

Figure 2.1: Annex-I greenhouse gas emissions excluding LULUCF between 1990 and 2007. Source: UNFCCC (2009).

Data on GHG emissions of Non-Annex-I countries is also available. However, most countries are reporting difficulties in measuring their greenhouse gas emissions. Therefore, it is not possible to obtain exact numbers for a specific year. The latest UNFCCC report of 2005 addresses this issue and presents numbers for "1994 or the closest year reported"[6]. GHG emissions of Non-Annex-I countries amount to approximately 11.74 bn tons of CO_2-e which corresponds to a share of about 40% in global GHG emissions. Moreover, the GHG emissions of Non-Annex-I countries have steadily increased. Nevertheless, the Kyoto Protocol only obliges Annex-I countries to reduce their GHG emissions. The rationale behind this is twofold.
First, the contribution of Non-Annex-I countries to the cumulative GHG emissions since the beginning of the Industrial Revolution is still relatively small. Second, Non-Annex-I countries argue that even today their emission level is low when taking into account that the population of Non-Annex-I countries is approximately four times larger than that of Annex-I countries. This means that the per capita emissions of Annex-I countries are roughly four times higher than the per capita emissions of Non-Annex-I countries.

[6]See UNFCCC (2005).

2.1c Permits for compliance with the Kyoto Protocol

The following section provides a survey on the different types of permits that are functional for compliance with the Kyoto Protocol.
Annex-I countries are obliged to establish a national registry recording the number of permits that a country holds. The UNFCCC Secretariat aggregates the data of the different national registries. The central registry for permits is called International Transaction Log (ITL).
The most important permit type is AAU (Assigned Amount Unit): The emissions in the baseline year 1990 determine the Assigned Amount, i.e. the total number of emission allowances that an Annex-I country receives for the compliance period (2008-2012). The Assigned Amount is equal to the emissions in the baseline year 1990 multiplied by a reduction factor and by five, the length of the compliance period in years. The reduction factor is equal to one minus the reduction target in per cent (e.g. the reduction factor for the aggregate Annex-I emissions is $100\% - 5.2\% = 94.8\% = 0.948$). One AAU (Assigned Amount Unit) gives the holder the right to emit one ton of CO_2-e during the compliance period. At the end of the compliance period an Annex-I country has to hand in one AAU per unit of emission (one ton of CO_2-e) to the regulator. Alternatively, an Annex-I country can also use permits from the flexible mechanisms up to a certain amount. If an Annex-I country fails to hand in a sufficient number of valid permits it faces a penalty for each unit of emission that is not covered by a permit.

The total number of AAU is fixed and cannot be altered. But the Emissions Trading mechanism of the Kyoto Protocol allows Annex-I countries to trade AAU among each other and thereby change the distribution of the AAU between the different national registries. Emissions trading of AAUs is subject to restrictions. Annex-I countries have to hold the so-called commitment period reserve (CPR) in their national registry. AAU can only be sold to other countries if after the transaction the selling nation still holds more AAUs in its national registry than the Commitment Period Reserve (CPR). For most developed countries the Commitment Period Reserve (CPR) is given by

$$\text{CPR} = 90\ \%\ \text{of Assigned Amount.}$$

The traded AAU might result from

1. Emission reduction efforts on a national level
   A country successful in reducing emissions below its Assigned Amount can consider selling parts of its AAU to other Annex-I countries. Conversely, an Annex-I country might realize that it is unable to achieve its reduction target solely by national measures. In such a case it might decide to buy AAU from other Annex-I countries.

2. Initial overallocation
   A country that was generously awarded permits (i.e. countries that have negotiated a favourable baseline) might sell those permits to other countries even though the permits simply originate from overallocation in this case. Especially, countries of the former Soviet bloc (EIT countries) have profited from the choice of the baseline year 1990. Economies of the EIT countries collapsed around 1990 and as a result of this collapse emissions dropped significantly and have not rebounded since then. Therefore, the Emissions Trading mechanism of the Kyoto Protocol has been criticized for creating "Hot Air" in the EIT countries.

Like the Emissions Trading mechanism, the JI mechanism (Joint Implementation) and the Clean Development Mechanism (CDM) do not alter the total number of AAU. But the JI mechanism and the CDM increase the total number of permits that are functional for compliance. Trading of permits from JI or CDM projects is also allowed. Those permits always result from an emission reduction project:

1. JI (Joint Implemenation)
   Emission reductions achieved by JI projects yield permits called ERU (Emission Reduction Unit). A JI project is an emission reduction project in an Annex-I country.

2. CDM (Clean Development Mechanism)
   Emission reductions achieved by CDM projects yield permits called CER (Certificates of emission reduction). A CDM project is an emission reduction project in a Non-Annex-I country, i.e. mainly in developing countries.

Furthermore, the number of permits in the national registry can be changed by LULUCF activities such as afforestation, reforestation and deforestation. LULUCF activities yield so-called Removal Units (RMU). Depending on the region where the LULUCF project is carried out the removal unit is referred to by a different name:

1. Domestic LULUCF projects in Annex-I countries yield RMU
2. LULUCF projects mainly in EIT countries yield ERU-RMU.
3. LULUCF activities in Non-Annex-I countries yield
   - tCER (temporary CER) or
   - lCER (long-term CER).

   The difference between tCER and lCER is due to the different project length and the way in which removals are accounted for. Annex-I countries are only allowed to use a limited number of tCER and lCER for compliance (up to 1% of their Assigned Amount).

Therefore, the number of permits in the national registry that are functional for compliance can be calculated as:

$$
\begin{aligned}
&\text{Number of permits in the national registry that are functional for compliance}\\
&\qquad = \quad \text{Assigned Amount}\\
&\qquad\quad \pm \text{AAU bought from/sold to other Annex-I countries}\\
&\qquad\quad \pm \text{ERU from JI projects carried out between 2008 and 2012}\\
&\qquad\quad + \text{CER from CDM projects carried out between 2001 and 2012}\\
&\qquad\quad \pm \text{RMU, ERU-RMU, tCER and lCER from LULUCF activities}
\end{aligned}
$$

The above formula allows us to determine whether an Annex-I country complies with its emission limits under the Kyoto Protocol. We have to compare the number of permits held in the national registry at the end of the compliance period with the emissions between 2008 and 2012. A country has achieved its reduction target if and only if

$$
\begin{array}{lcl}
\text{Number of permits in the national registry} & \geq & \text{National Annex A emissions}\\
\text{on 31 March 2015} & & \text{between 2008 and 2012.}
\end{array}
$$

Due to the relatively complex reporting process the compliance date (31 March 2015) is more than two years after the end of the first commitment period of the Kyoto Protocol (2008 – 2012).

Regulations of the following two cases are of special interest:

1. Non-compliance
   The Annex-I country has not enough permits in the national registry
2. Over-compliance
   The Annex-I country has more permits in the national registry than it needs for compliance

In case of non-compliance, the Annex-I country faces a penalty. A non-compliant country has to reduce emissions in the next compliance period by the additional amount of

$$1.3 \cdot \max(\text{Annex A emissions between 2008 and 2012 - Permits in the national registry}; 0)$$

tons of carbon dioxide equivalent. This means that the Annex-I country is not released from the obligation to hand in the lacking permit in the second compliance period (after 2012). On the contrary, it has to hand in 0.3 additional permits in the course of the second compliance period. However, the penalty will only affect non-compliant countries if there is a second compliance period. As political leaders failed to negotiate a Post-Kyoto Protocol so far, it is uncertain at the moment whether there will be a second compliance period.

In case of over-compliance, the Kyoto Protocol regulates the bankability of permits from the current Kyoto Protocol compliance period (2008-2012) to the next compliance period (after 2012). This means that an Annex-I country is allowed to transfer surplus permits of the current compliance period into the following compliance period. Table 2.4 provides an overview of the permits that are functional for compliance with the Kyoto Protocol and of the rules regarding the bankability of the permits.

The signatories to the Kyoto Protocol implicitly assumed that there will be a Post-Kyoto Protocol with a second compliance period. The existence of a second compliance period is crucial because otherwise there will be no penalty and Annex-I countries that reduced emissions will not be able to profit from the possibility of banking permits. Twelve years after the adoption of the Kyoto Protocol optimistists expected that the Copenhagen Conference in December 2009 would finally yield an agreement on the second compliance period. However, the Copenhagen Conference ended without a binding result.

Name of permit	Reduction project	Bankability of permits
AAU	Reduction of Annex A emissions[7] in an Annex-I country	Yes
ERU	Reduction of Annex A emissions in another Annex-I country	Yes, but limited
CER	Reduction of Annex A emissions in a Non-Annex-I country	Yes, but limited
RMU and ERU-RMU	LULUCF activity in an Annex-I country	No
lCER and tCER	LULUCF activity in a Non-Annex-I country	No

Table 2.4: Overview of banking regulations for permits that are eligible for compliance with the Kyoto Protocol.

2.1d Post-Kyoto Protocol

The negotiations during the Conference in Copenhagen (also known as COP-15) in December 2009 mainly focused on the following issues

- Target of limiting global warming to 2 °C compared to pre-industrial levels
- Reduction commitments of developed countries
- Commitment of emerging economies such as China and India to cut their emission intensity[8]
- Monitoring of the GHG emissions in developing countries and emerging economies by an international agency
- Financial assistance for developing countries in adapting to climate change

Participants of the Copenhagen Conference did not manage to reach a binding agreement on any of the above items. A minor success of the Copenhagen Conference is that a group of countries including China, India and Brazil recognize the need for limiting global

[7] The definition of Annex A emissions can be found in Section 2.1b.
[8] Emission intensity is defined as GHG emissions per unit of GDP (Gross Domestic Product).

warming to $2\,^{\circ}\mathrm{C}$. However, this accord has a very weak legal status as pointed out by China's chief negotiator: "This is not an agreed accord, it is not an agreed document, it is not formally endorsed or adopted"[9].

According to the analysis of U.S. Energy Information Administration (2009), China and the US are responsible for about 40% of today's global GHG emissions (both China and the US account for roughly 20%). The European Union is the third largest emitter with a share of approximately 15%. During the Copenhagen Conference the three largest emitters announced the targets listed in Table 2.5.

Emitter	**Reduction of**	**Baseline year**	**Reduction target**
EU	Emissions in 2020	1990	20%
US	Emissions in 2020	2005	17%
China	Emission intensity in 2020	2005	40-45%

Table 2.5: Reduction commitments of the three largest GHG emitters (announced in December 2009). Source: Harvey (2009).

The European Union is the only emitter with an ambitious reduction target[10]. The EU announced a target that reduces emissions both below the reduction target of the Kyoto Protocol (-8% compared to the baseline year 1990) and below current emission levels. The United States intend to reduce emissions below the current emission level. However, the reduction target announced during the Copenhagen Conference in December 2009 is less ambitious than the reduction target of the Kyoto Protocol (-7% compared to the baseline year 1990) to which the US committed themselves in 1997. As the development of the GDP has to be taken into account, China's final reduction target might differ.

[9] See Harvey et al. (2009).

[10] This follows from the emissions data and the estimates of Crooks and Romei (2009), Global Carbon Project (2008), Harvey (2009), Pew Center on Global Climate Change (2009), UNFCCC (2008), UNFCCC (2009), U.S. Energy Information Administration (2009).

2.2 EU ETS (EU Emissions Trading Scheme)

2.2a Introduction

With the ratification of the Kyoto Protocol the European Union committed themselves to an ambitious reduction target. In 2005 the EU launched the first international CO_2 emissions trading system (in the following denoted by EU ETS) in order to achieve its reduction target. The EU ETS covers more than 10,000 large installations in the energy sector and in several industrial sectors which emit approximately half of the CO_2 emissions of the European Union. Covering roughly 40 per cent of all GHG emissions in the EU, the EU Emissions Trading Scheme (EU ETS) has become the cornerstone of environmental politics in Europe. Since 2008 the EU ETS has covered all EU-27 countries[11] plus Norway and Liechtenstein.
The main difference between the EU ETS and the Emissions Trading mechanism of the Kyoto Protocol is that trading takes place between companies and not between countries. Furthermore, in contrast to the Emissions Trading mechanism of the Kyoto Protocol, the EU ETS is a very liquid market. Therefore, the majority of the academic papers focus on the price dynamics of the EU ETS.

The EU ETS comprises several compliance periods:

- Phase I (2005-2007)
- Phase II (2008-2012)
  It is important to note that the second phase of the EU ETS coincides with the first compliance period of the Kyoto Protocol (2008-2012).
- Phase III (2013-2020)

At the beginning of each phase the regulator hands out permits to the regulated companies. The number of permits for each installation is defined in the National Allocation Plan (NAP) approved by the European Commission (EC). Permits of the EU ETS are referred to by EUA (European Union Allowances). The owner of an EUA is allowed to emit one ton of CO_2-e .
The main method of allocation during both Phase I and Phase II has been grandfathering which means that allowances are allocated free of charge. Furthermore, the number

[11] We refer to Section 2.3a for a list of all EU-27 countries.

of allowances that the regulator allocates to the regulated companies is related to their historical emissions. Member states of the European Union are only allowed to auction up to 5% of all permits during Phase I and up to than 10% during Phase II.
Similar to the Emissions Trading mechanism of the Kyoto Protocol, regulated companies have to hand in one permit per emitted unit of carbon dioxide. In case of non-compliance a company has to pay a penalty for every unit of emission that is not covered by an emission permit. Moreover, the payment of the penalty does not release the regulated company from the obligation to hand in the lacking permits. The penalty is 40 Euro per ton of CO_2-e during the first phase and 100 Euro per ton of CO_2-e during the second phase of the EU ETS[12].

The legal framework for Phase I and II of the EU ETS is based on Directive 2003/87/EC. The directive has been transformed into German law by the adoption of the "Treibhaus-Emissionshandelsgesetz (TEHG)" on 8 July 2004 and the "Gesetz über den Nationalen Zuteilungsplan für Treibhausgas-Emissionsberechtigungen in der Zuteilungsperiode 2005 bis 2007 – Zuteilungsgesetz 2007 (ZuG 2007)" on 26 August 2004. The German Bundestag adopted the German National Allocation Plan for Phase II by enacting the law "Gesetz über den Nationalen Zuteilungsplan für Treibhausgas-Emissionsberechtigungen in der Zuteilungsperiode 2008 bis 2012 – Zuteilungsgesetz 2012 (ZuG 2012)" on 7 August 2007.

Figure 2.2 and Table 2.6 provide a first insight into the structure of the European permit market. Figure 2.2 shows that the EU ETS is mainly dominated by the combustion sector and the three following industries: cement, iron and steel and refineries. The rest of 5% consists of the following industries: paper (1.8%), coke ovens (1.1%), glass (1.1%), ceramics (0.9%) and metal ore (0.4%). It is remarkable that installations with annual emissions of more than 1 m tons CO_2-e are responsible for roughly two thirds of the overall emissions in the EU ETS whereas the aggregate emissions of small installations (with emissions of less than 0.01 m tons CO_2-e) are negligible. Their share in total EU ETS emissions amounts to less than 8% (cf. Table 2.6).

[12] A detailed description of the compliance regulations is to be found in Section 2.2c.

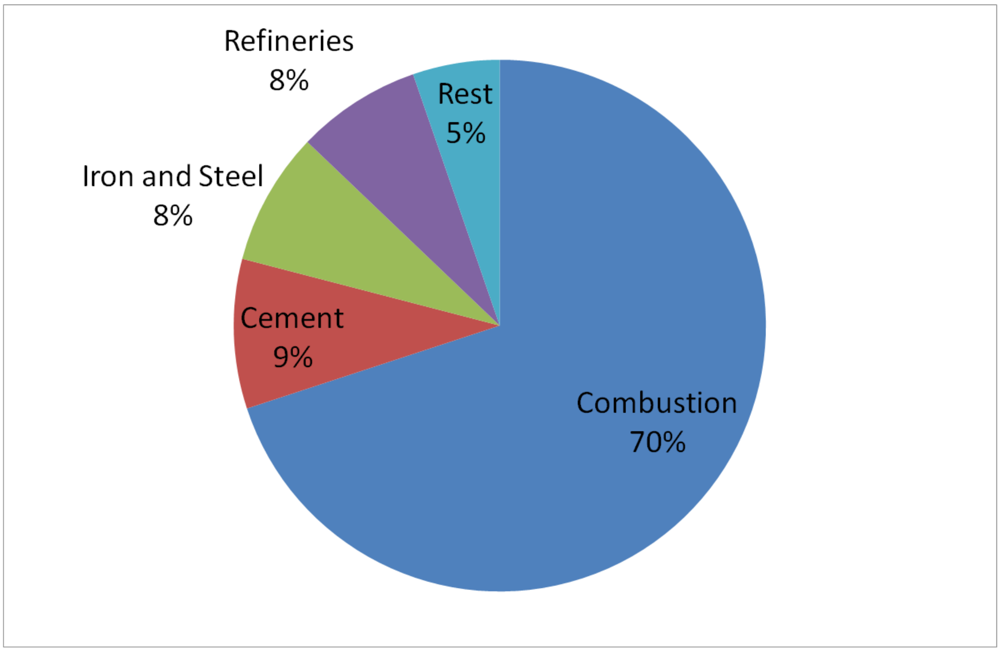

Figure 2.2: Allowance allocation to different EU ETS sectors in the year 2006.
Source: Analysis of CITL data by Trotignon and Delbosc (2008).

2.2b Baseline

The following paragraph contains a more detailed survey on the procedure of setting the emission cap for a compliance period. Member states of the European Union have to compile a National Allocation Plan (NAP) for each compliance period consisting of a macro plan and a micro plan. The NAP for the second phase of the EU ETS (2008-2012) was due in summer 2006 and was approved by the EC at the end of 2006. The macro plan defines a maximum amount of national emissions during the compliance period. This

Size of installation (emissions in 1,000 tons of CO2-e)	**Share in total emissions**
Less than 10	1%
10 – 100	7%
100 – 500	12%
500 – 1,000	12%
More than 1,000	68%

Table 2.6: Breakdown of EU ETS emissions according to the size of the installations
Source: Analysis of CITL data by Trotignon and Delbosc (2008).

amount must be in line with the emission reduction obligation under the Kyoto Protocol (Burden Sharing Agreement). Finally, the macro plan defines how the total emission budget is distributed between installations covered by the EU ETS (cf. Figure 2.2 and Table 2.6) and between the remaining emitters. The splitting up of the emission budget is performed in four steps (cf. Figure 2.3):

1. Size of the total emission budget
   is obtained by analyzing historical GHG emissions (nationwide data).

2. Size of CO_2 emissions budget
   The total emission budget is split up into CO_2 emissions and non- CO_2 emissions. The EU ETS almost exclusively covers CO_2 emissions. During the first phase (2005-2007) the EU ETS only covered CO_2 emissions. In Phase II (2008-2012), however, some other greenhouse gases have been included, e.g. nitrous oxide emissions from Dutch and Norwegian installations that produce nitric acid.

3. Size of CO_2 emissions budget for the energy sector and the industry
   According to a detailed analysis CO_2 emissions are distributed among private households, traffic and the energy sector and the industry.

4. Cap in the EU ETS
   The plan defines how much of the CO_2 emissions budget of the energy sector and of the industry is allocated to installations covered by the EU ETS. We refer to this by German cap in the EU ETS.

The micro plan breaks the cap of the macro plan down to the level of installations and specifies how many allowances are allocated to each installation.

The EC corrected the German emission cap in the EU ETS from 482 m tons CO_2-e to 453 m tons CO_2-e . This corresponds to a reduction of roughly 6 per cent. Germany was not the only country whose NAP was modified by the EC. The analysis of Point Carbon (2008) shows that the revised NAPs for Phase II allocate roughly 2,100 m tons CO_2-e to all installations covered by the EU ETS. In total these are 10.4 per cent fewer permits than foreseen in the NAPs submitted to the EC. This means that the emission cap of the revised NAPs is approximately 9% below the “business as usual” (BAU) scenario. In

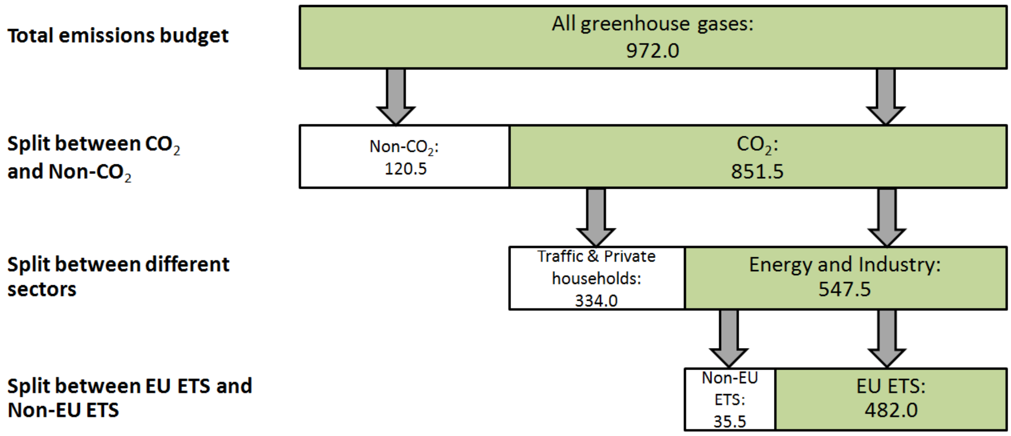

Figure 2.3: Breakdown of the German emission budget (in m tons CO2-e) according to the macro plan in the NAP 2008-2012.
Source: Bundesministerium für Umwelt (2006).

other words, the NAPs submitted to the EC would have allowed regulated companies on average even 1-2% more emissions than in the "business as usual" (BAU) scenario. Annex II of Directive 2003/87/EC lists twelve criteria an NAP has to fulfil. It is upon the EC to check the NAPs for validity and the EC has the right to modify the NAP if some of the criteria are violated. Poland sued the EC for rejecting its NAP without providing an explanation consistent with Annex II of Directive 2003/84/EC. The EC demanded a reduction of Poland's allowances by more than 25 per cent[13]. On 23 September 2009 the Court of First Instance annulled the decision of the EU. This event immediately drove the permit price four per cent lower because the overall cap of the EU ETS was expected to be loosened. If Poland will finally succeed in being allocated a relatively large share of permits to its installations covered by the EU ETS, the confidence in the EU ETS might be reduced. Point Carbon, however, assesses the probability of significantly lower permit prices in the wake of an approved Polish NAP to be low. They argue that the bankability of permits between Phase II and Phase III together with a presumed permit shortage in Phase III will prevent prices from falling too low[14]. The issue with the Polish NAP shows, however, that emission trading systems are surrounded by regulatory uncertainty - even in the course of a compliance period the number of allowances in the system could be adjusted.

[13] See European Union (2007).
[14] See Morrison (2009).

2.2c European Union Allowance (EUA)

In the following we provide a detailed survey on the main characteristics of European Union Allowances (EUA). Taking into account that a permit is always related to a certain compliance period we denote an EUA of the first phase by EUA1. Similarly, an EUA of the second and of the third compliance period are denoted by EUA2 and EUA3, respectively. National registries document changes in the number of allowances held by the installations. On a European level data is aggregated by the Community Independent Transaction Log (CITL).
EU ETS installations have an annual compliance obligation. Permits must be handed in to the regulator by 30 April of the following year (e.g. the date for 2009 compliance is 30 April 2010). In case of non-compliance a company has to pay a penalty for every unit of emission that is not covered by an emission permit. Moreover, the payment of the penalty does not release the regulated company from the obligation to hand in the lacking permits in the following calendar year. The penalty is 40 Euro per ton of CO_2-e during the first phase and 100 Euro per ton of CO_2-e during the second phase of the EU ETS. Due to the permission of intra-phase banking and intra-phase borrowing it is unlikely that a company will not be able to comply in a year prior to the final year of the compliance period. In other words, the EU ETS is very similar to a system where there is a compliance obligation only at the end of a compliance period.
Intra-phase banking means that a company can use certificates that it did not use up in the year before for compliance in the following year. This might be the case if a company emitted less CO_2 than it was allowed to.
Conversely, a company that emitted more CO_2 than it was allowed to might use emission permits of the following year for compliance purposes in the current year. This process is called intra-phase borrowing.

The design of the EU ETS strictly forbids inter-phase borrowing. Inter-phase banking was not allowed from Phase I (2005-2007) to Phase II (2008-2012) but this regulation has been changed and now inter-phase banking is allowed from Phase II (2008-2012) to Phase III (2013-2020).
Inter-phase borrowing means that companies are allowed to use EUA of the following phase for compliance in the current phase. The reason why inter-phase borrowing is not allowed is that it would significantly reduce the incentives for emission reductions and therefore put the environmental targets of the regulator at risk.
Inter-phase banking occurs when a company transfers permits from a previous compliance period into the current compliance period. The permission of inter-phase banking does

not put environmental targets at risk.

In case that there should be an over-allocation of permits in one phase, however, the inter-phase banking would also effect the following periods. Over-allocation describes a situation in which the regulator hands out more permits to the regulated companies than they need. This might happen when an emission trading system is introduced (regulator might not be able to estimate the baseline correctly or regulator does not want to scare regulated companies and therefore allocates permits generously) or also during periods of recession (when decision on NAPs for the coming period is taken during a recession and regulator does not want to put further pressure on the industry). Table 2.7 summarizes the main characteristics of Phase I (2005-2007) and Phase II (2008-2012) of the EU ETS.

Phase	**Name of permit**	**Penalty**	**Intra-phase banking and borrowing**	**Inter-phase banking**	**Inter-phase borrowing**
Phase I (2005-2007)	EUA1	40 EUR	Yes	No	No
Phase II (2008-2012)	EUA2	100 EUR	Yes	Yes	No

Table 2.7: Survey on the first two phases of the EU ETS.

2.2d CER (Certificate of Emission Reduction) and ERU (Emission Reduction Unit)

Link between Kyoto Protocol and EU ETS

Directive 2004/101/EC (also known as linking directive) sets up the link between the EU ETS and the Kyoto Protocol. It is crucial to bear in mind that the Kyoto Protocol establishes emission trading of AAU between different countries. Furthermore, the Kyoto Protocol sets up the CDM and JI mechanisms which yield permits that can be traded by countries and by private institutions.
However, the EU ETS focuses on emissions trading between regulated companies in the European Union.

In Phase I of the EU ETS regulated companies were not allowed to use CER and ERU for compliance. This has changed in Phase II of the EU ETS. The only exception concerns CER and ERU from LULUCF activities. tCER, lCER and ERU-RMU are not eligible for compliance with the EU ETS. Table 2.8 provides an overview of the eligibility of permits for compliance with both the Kyoto Protocol and the EU ETS.

Name of permit	**Eligible for compliance with Kyoto Protocol**	**Eligible for compliance with EU ETS**
EUA	No	Yes
AAU	Yes	No
CER	Yes	Yes[15]
ERU	Yes	Yes[15]
RMU	Yes	No
tCER, lCER	Yes	No
ERU-RMU	Yes	No

Table 2.8: Survey on the eligibility of different permits for compliance with the Kyoto Protocol and the EU ETS.

Detailed rules on the use of CER/ERU for compliance in the EU ETS

As in the case of EUA one has to distinguish between the different compliance periods[16]. Therefore, the following notation is introduced: KP-1 CER and KP-1 ERU denote permits that are functional for compliance in the first Kyoto Protocol compliance period, i.e. in the second EU ETS compliance period. KP-2 CER and KP-2 ERU can be used for compliance in the second Kyoto Protocol compliance period, i.e. the third EU ETS compliance period. Due to possible delays in the verification process of emission reductions a KP-2 CER/ERU might also result from an emission reduction that took place before 2013. Table 2.9 provides an overview of the different permits from CDM and JI projects.

It is important to understand the regulations on the use of CER and ERU for compliance purposes with the EU ETS in detail. There is a formal difference between EUA and

[15] No in Phase I of the EU ETS (2005-2007).

[16] In the case of CER and ERU we refer to a Kyoto Protocol compliance period and not to an EU ETS compliance period. The first Kyoto Protocol compliance period and the second EU ETS compliance period are both 2008-2012.

Name of certificate	Compliance period	Year of emission reduction
KP-1 CER	2008 – 2012	Nov 2001 – Dec 2012
KP-1 ERU	2008 – 2012	Jan 2008 – Dec 2012
KP-2 CER and KP-2 ERU from pre-2013 projects	2013 – 2020	Jan 2008 – Dec 2012
KP-2 CER and KP-2 ERU from post-2013 projects	2013 – 2020	Jan 2013 – Dec 2020

Table 2.9: Classification of certificates from CDM and JI projects.
Source: Curien and Lewis (2009).

CER/ERU regarding compliance. CER/ERU are only de facto compliance instruments in Phase II of the EU ETS. CER/ERU, however, are not de iure compliance instruments. This has to do with the specific procedure. When a CER/ERU is handed in for compliance purposes the relevant member state immediately issues an EUA against this CER/ERU, with the EUA then being subsequently cancelled. Thus the EUA effectively never comes into circulation. Those subtle differences and the bankability regulations for EUA and CER/ERU are summarized in Table 2.10.

	Eligibility of permits for compliance in	
Name of certificate	**Phase II of EU ETS (2008-2012)**	**Phase III of EU ETS (2013-2020)**
EUA1	No (banking forbidden)	No (banking forbidden)
EUA2	Yes - de iure	Yes - de facto
KP-1 CER/ERU	Yes - de facto	Yes - de facto
EUA3	No (borrowing forbidden)	Yes - de iure
KP-2 CER/ERU from pre-2013 projects	No (borrowing forbidden)	Yes - de facto
KP-2 CER/ERU from post-2013 projects	No (borrowing forbidden)	Yes - de facto but only if project is registered in a Least Developed Country (LDC)

Table 2.10: Eligibility of different permits for compliance with the EU ETS in Phase II and III. Source: Curien and Lewis (2009).

There are restrictions on the use of CER/ERU for compliance purposes in the EU ETS. The exact rules have been updated several times. The limits defined in the "Rulings of the European Commission on Phase-2 NAPs of the member states" were tightened in the ETS review process in January 2008 and finally relaxed again in December 2008 (cf. Table 2.11). EU ETS installations used 82 m tons CO_2-e of CER and ERU[17] for compliance in 2008 which corresponds to a share of approximately 4% in the total number of permits handed in for compliance. This amount is below the limit (cf. Table 2.11). This follows immediately from

$$\begin{aligned} 1638 \text{ m tons } CO_2\text{-e/Length of Phase II and III} &= 1638 \text{ m tons } CO_2\text{-e/13 years} \\ &= 126 \text{ m tons } CO_2\text{-e per year} \\ &> 82 \text{ m tons } CO_2\text{-e per year.} \end{aligned}$$

Name of the decision defining the limit	**Maximum amount of CER/ERU (in m tons CO_2-e) eligible for compliance**
Rulings of the European Commission on Phase-2 NAPs of the member states	1390 for Phase II[18]
ETS review process of the European Commission in January 2008	1390 for Phase II and III together
Revised ETS Directive agreed on at the EU Heads of Government Summit in December 2008	1638–1885 for Phase II and III together[19]

Table 2.11: Restrictions on the use of CER/ERU for compliance purposes in the EU ETS.

[17] See European Union (2009). The importance of ERUs is negligible - allowances for 0.04 m tons CO2-e were handed in for compliance in 2008.

[18] This limit corresponds to 13.3% of the aggregate Phase-2 EUA allocation. However, the limit varies from country to country ranging from 0% (Estonia) to 22% (Germany).

[19] Estimation of Curien and Lewis (2009) - the precise amounts will be decided under the ongoing Comitology process.

2.2e Verified emissions

Annex IV of Directive 2003/87/EC specifies the principles for monitoring and reporting emissions. Regulated companies are either obliged to

- measure emissions of their installations or
- calculate emissions using activity data (fuel used, production rate, etc.), emission factors and oxidation factors.

Data must be verified by an independent auditor and then be sent to the regulator. The regulator checks and aggregates the data. Verified emissions are published by the EC in May or June of the following year (cf. Table 2.12).

Press release	**Date**	**Verified emissions of**
IP/06/612	15 May 2006	2005
IP/07/776	7 June 2007	2005, 2006
IP/08/787	23 May 2008	2005, 2006, 2007
IP/09/794	15 May 2009	2008

Table 2.12: Press releases on verified EU ETS emissions.
Source: European Union (2006, 2007, 2008a, 2009).

Table 2.18 and 2.19 show that it was impossible to determine verified emissions for all installations in 2005. The Czech Republic, France, the Slovak Republic and Spain were reporting technical problems with their national registries. Moreover, data for Cyprus, Luxembourg, Malta and Poland was completely missing. Reporting has been improved since then. However, there are still updates on past verified emission data.

In addition to possible updates, one has to bear in mind that the number of covered installations changes over time and therefore, data has to be interpreted carefully. Therefore, one cannot use the numbers in Table 2.18 and 2.19 for assessing whether EU ETS installations reduced their emissions.
Using verified emissions in 2005 as a baseline and adjusting for changes in the number of installations covered by the EU ETS, the European Union comes to the conclusion that emissions slightly increased by 0.7% during the first compliance period (2005-2007). The number of installations in Phase II significantly differs from the number of installations in Phase I. Therefore, it is not possible to assess the change in emissions from 2007 to

2008. As long as there are significant changes in the baseline (i.e. in the set of installations covered by the EU ETS) it is very difficult to find out to which degree the EU ETS incentivized regulated companies to reduce emissions.

2.2f Historical permit prices

There is both, a spot and a futures market for EUA and CER. However, the volume of the futures market is by far larger than that of the spot market[20].

In 2008 emission allowances were traded mostly over the counter (60%) and on several exchanges (40%). The share of the exchange traded volume steadily increased since the launch of the EU ETS[21]. Exchanges for EUA and CER:

- European Climate Exchange (ECX) in London
  is by far the most important exchange with a market share of 87%
- Bluenext in Paris
- NordPool in Oslo
- EEX in Leipzig
- EXAA in Vienna

Permit prices in newspapers refer to either the spot price or to a trade-weighted average of all trades on the futures market (denoted by Emissions Index). However, the most liquid contracts are the futures contracts. A futures contract is an agreement to buy or sell an asset at a certain future time (referred to by maturity of the futures contract) for a certain price. With entering into a long/short EUA-Dec10 futures contract on e.g. 30 November 2009 the party agrees to buy/sell one EUA in December 2010 at the EUA-Dec10 futures price quoted on the exchange on 30 November 2009. The EUA-Dec10 contract is said to mature in December 2010. On 30 November 2009, the following futures contracts were traded on the ECX: EUA-Dec09, EUA-Dec10, EUA-Dec11 and EUA-Dec12. Among those contracts, the EUA-Dec09 contract is the one that matures next and therefore, on 30 November 2009 it is called front contract. From January to December 2010, the front contract will be EUA-Dec10.

[20] See ECX (2009).

[21] See Point Carbon (2008) and Point Carbon (2009).

Figure 2.4 - 2.6 show that the front contract has always the largest market share. The only exception is the year 2007 where the EUA-Dec08 futures contract (maturing in Phase II) is leading instead of the EUA-Dec07 futures contract (maturing in Phase I). The significance of the front contract is much more pronounced for EUA than for CER. The market share of the EUA front contract was 60-90% compared to 30-60% for the CER front contract.

Permit prices exhibited the following characteristics:

- The spot permit price and also the EUA-Dec07 futures price converged to zero at the end of the first compliance period (cf. Figure 2.7).
- The relationship between the spot price and prices of futures that mature in the current compliance period can be described by a cost-of-carry relationship as shown by Uhrig-Homburg and Wagner (2007). This can also be observed graphically in Figure 2.7 - 2.9.
- The price relationship between futures that mature in different compliance periods is much more complex (cf. Figure 2.7). The bankability of permits from the current to the next phase heavily influences this relationship. If banking of permits is not allowed a completely different price behavior of those two futures contracts is most likely (cf. Figure 2.7). Even if banking of permits is allowed prices might differ significantly at the end of the compliance period. The reason is that the price of the futures contract maturing in the current compliance period reflects expectations of a permit shortage in the current compliance period. This information is not included in the price of the futures contract maturing in the next compliance period.
- Permit prices are very volatile and exhibit a jumpy behaviour.

Futures contracts for CER were launched on the ECX in 2008. Some authors denote this type of contract primary CER (pCER) in order to distinguish it from a so-called secondary CER (sCER). pCER are mainly traded on an exchange and do not involve a counter-party risk, i.e. the delivery of the emission allowance is ensured. sCER are mainly traded over the counter and contain a counter-party risk. Therefore, sCER are cheaper than pCER. In the following we focus on pCER and refer to them simply by CER.
The CER price is following the EUA price and CERs are always cheaper than EUAs (cf. Figure 2.10 - 2.11). Possible explanations are that the costs for reducing emissions in developing countries (yielding CER) are much lower than the abatement costs in Europe (yielding EUA). Furthermore, one could argue that the EU ETS has a clear enforcement

structure whereas developing countries have not committed themselves to any emission reduction. Without the existence of emission trading systems such as the EU ETS permits such as CERs are almost worthless. As the EU ETS dominates the global carbon market most of the CER are used for compliance in the EU ETS and therefore the EUA price should be an upper bound for the CER price.

2.2g Outlook on Phase III (2013-2020)

The third EU ETS phase is planned between 2013 and 2020. Directive 2009/29/EC of 23 April 2009 is the legal basis for the third phase. Its contents[22] have been agreed on by EU leaders and the European Parliament in December 2008. With the adoption of Directive 2009/29/EC the European Union commits itself to reduce its GHG emissions by 20%. compared to the 1990 emission level. This aim should be achieved by 2020. In case there should be a Post-Kyoto Protocol, the EU intends to reduce GHG emissions even by 30%. In addition to the overall target, the EU also specified targets for the industries covered by the EU ETS. These industries have to reduce their emissions by 21% compared to the verified emissions of the year 2005. This corresponds to an additional reduction of 12% compared to the level of 2012.

There are several changes in the regulations compared to those of Phase II:
First, in addition to the industries that are already covered by the EU ETS, the following sectors will be included from Phase III on:

- Aviation, petrochemicals, ammonia, aluminium
- Carbon Capture and Storage (CCS)
  Directive 2009/29/EC reserves 300 million allowances for twelve commercial scale demonstrations of Carbon Capture and Storage, thus co-funding those CCS projects.

Moreover, there are changes in the regulations concerning the covered greenhouse gases and the size of the regulated installations:

- Inclusion of N_2O and perfluorocarbons emissions (mainly from the ammonia and aluminium sector).

[22]We refer to European Union (2008b) and to the summary of the Department of Energy & Climate Change in the UK (2008) for a comprehensive overview.

- Exclusion of smaller installations.
  An installation that emits less than 25,000 t of CO_2-e per year or a combustion installation with a capacity of less than 35 MW is defined as small.

Because of the changes described above, the baseline for Phase II and III cannot be compared to each other directly. Especially, absolute figures on the amount of allowances have to be treated carefully as one has to know about the set of sectors to which they refer.

Second, an allocation plan on a European scale will replace National Allocation Plans. This will reduce the regulatory uncertainty (cf. Section 2.2b).

Third, there will be a substantial increase in the amount of allowances that are auctioned. In 2020 the minimum share of auctioned permits will be 60% compared to 3% in Phase II. However, the rules of auctioning are not the same for the energy sector and the industrial sectors. The energy sector straight through the European Union, with the exception of some new member states, will have to cope with an auction level of 100%. The minimum rate for the energy sector in the new member states will rise from 20% in 2013 to 70% in 2020.
Within the industrial sector the EC categorizes installations with respect to their risk of carbon leakage. Carbon leakage describes the probable effect that industries emitting a lot of carbon-dioxide and unable to reduce their emissions significantly will face huge costs (being forced to buy permits from the other installations) and therefore might probably transfer their production to countries without a GHG reduction obligation. 90% of all industrial installations are categorized as having a “high risk of carbon leakage”. Industries without a high risk of carbon leakage will have auction levels between 20% (2013) and 70% (2020).

Fourth, the regulator tightened the limits on the use of CER and ERU for compliance purposes in Phase III. The maximum amount of permits originating from outside the EU is equal to 50% of the required emission reductions in the EU.

Fifth, in case that permit prices are rising too high, the regulator will auction parts of its New Entrants Reserve. The intention of the so-called allowance reserve mechanism is to reduce permit prices - at least for a while. The upper price barrier that triggers the regulator’s market intervention varies over time and is defined as the average permit price of the preceding two years multiplied by three.

To conclude, the designs of Phase II and Phase III of the EU ETS differ significantly and the exact regulations for Phase III are still subject to changes until 2013. Phase III will begin in 2013.

It is important to note that Phase III will be implemented even without the adoption of a Post-Kyoto Protocol. However, the permit price drop of -10% on the day after the end of the Copenhagen Conference highlights the fact that uncertainty about the future of the Kyoto Protocol also affects the EU ETS. This can be explained as follows: The EU ETS is the main instrument of the European Union to achieve its emission reduction target of the Kyoto Protocol. Therefore, uncertainty about future reduction targets of other large emitters might decrease the willingness of the European Union to keep its own ambitious reduction targets.

2.3 Tables and graphs

2.3a Signatories of the Kyoto Protocol

- **Annex-I countries** include the industrialized countries that were members of the OECD in 1992, plus the EIT countries. In other words the list of Annex II countries plus the list of EIT countries is identical with the list of Annex-I countries. Annex-I countries have emission-reduction targets. These targets are listed in Annex B to the Kyoto Protocol.
- **EIT countries (Economies In Transition)** are 10 nations in mainly Eastern Europe. An EIT country is also in the list of Annex-I countries.
- **Annex-II countries** are those Annex-I countries that were members of the OECD in 1992.
- **Non-Annex I countries** are countries that are not in the Annex I list. Therefore, those countries have no emission-reduction targets.
- **LDC countries** are considered by the United Nations as the least developed countries. Most of the LDC countries are located in Africa or Asia.

List of Annex-I countries			
Australia	Finland	Lithuania	Slovenia
Austria	France	Luxembourg	Spain
Belarus	Germany	Monaco	Sweden
Belgium	Greece	Netherlands	Switzerland
Bulgaria	Hungary	New Zealand	Turkey
Canada	Iceland	Norway	Ukraine
Croatia	Ireland	Poland	UK
Czech Republic	Italy	Portugal	USA
Denmark	Japan	Romania	
Estonia	Latvia	Russian Federation	
European Community	Liechtenstein	Slovakia	

Table 2.13: List of Annex-I countries. Source: UNFCCC.

List of EIT countries			
Belarus	Estonia	Romania	Ukraine
Bulgaria	Latvia	Russian Federation	
Croatia	Lithuania	Slovenia	

Table 2.14: List of EIT countries. Source: OECD.

List of EU-15 countries			
Austria	France	Italy	Spain
Belgium	Germany	Luxembourg	Sweden
Denmark	Greece	Netherlands	UK
Finland	Ireland	Portugal	

Table 2.15: List of EU-15 countries. Source: European Union.

List of EU-27 countries			
Austria	Finland	Latvia	Romania
Belgium	France	Lithuania	Slovakia
Bulgaria	Germany	Luxembourg	Slovenia
Cyprus	Greece	Malta	Spain
Czech Republic	Hungary	Netherlands	Sweden
Denmark	Ireland	Poland	UK
Estonia	Italy	Portugal	

Table 2.16: List of EU-27 countries. Source: European Union.

List of LDC countries			
Africa			
Angola	Eritrea	Mauritania	Uganda
Benin	Ethiopia	Mozambique	United Republic of Tanzania
Burkina Faso	Gambia	Niger	Zambia
Burundi	Guinea	Rwanda	
Central African Republic	Guinea-Bissau	São Tomé and Principe	
Chad	Lesotho	Senegal	
Comoros	Liberia	Sierra Leone	
Democratic Republic of the Congo	Madagascar	Somalia	
Djibouti	Malawi	Sudan	
Equatorial Guinea	Mali	Togo	
Asia			
Afghanistan	Kiribati	Nepal	Tuvalu
Bangladesh	Laos	Samoa	Vanuatu
Bhutan	Maldives	Solomon Islands	Yemen
Cambodia	Myanmar	Timor-Leste	
Latin America and the Caribbean			
Haiti			

Table 2.17: List of LDC countries. Source: UN-OHRLLS.

2.3b Detailed analysis of EU ETS emissions data

Emissions	2005			2006		2007		2008
Press Release	**06**	**07**	**08**	**07**	**08**	**08**	**09**	**09**
Austria	33	33	33	32	32	32	32	32
Belgium	55	55	55	55	55	53	53	55
Cyprus	-	5	5	5	5	5	5	6
Czech Republic	82	82	82	84	84	88	88	80
Denmark	26	26	26	34	34	29	29	27
Estonia	13	13	13	12	12	15	15	14
Finland	33	33	33	45	45	43	43	36
France	131	131	131	123	127	127	127	123
Germany	474	475	475	478	478	487	487	473
Greece	71	71	71	70	70	73	73	70
Hungary	26	26	26	26	26	27	27	27
Ireland	22	22	22	22	22	21	21	20
Italy	215	226	226	227	227	226	226	221
Latvia	3	3	3	3	3	3	3	3
Lithuania	7	7	7	7	7	6	6	6
Luxembourg	-	3	3	3	3	3	3	2
Malta	-	-	-	-	-	-	2	2
Netherlands	80	80	80	77	77	80	80	84
Poland	-	202	203	209	210	210	210	204
Portugal	36	36	36	33	33	31	31	30
Romania	-	-	-	-	-	-	70	64
Slovak Republic	25	25	25	26	26	25	25	25
Slovenia	9	9	9	9	9	9	9	9
Spain	181	184	184	179	180	186	187	163
Sweden	19	19	19	20	20	15	19	20
UK	242	242	243	251	251	257	257	265
Σ	**1,785**	**2,011**	**2,012**	**2,027**	**2,034**	**2,050**	**2,126**	**2,060**
Bulgaria	-	-	-	-	-	-	-	38
Liechtenstein	-	-	-	-	-	-	-	-
Norway	-	-	-	-	-	-	-	19

Table 2.18: Verified EU ETS emissions in m tons CO_2-e according to the different press releases. Source: European Union (2006, 2007, 2008a, 2009).

Emissions	**2005**			**2006**		**2007**		**2008**
Press Release	**06**	**07**	**08**	**07**	**08**	**08**	**09**	**09**
Austria	199	199	199	197	197	210	210	216
Belgium	309	309	309	309	309	309	309	302
Cyprus		13	13	13	13	13	13	13
Czech Republic	389	395	395	405	405	406	406	401
Denmark	380	380	380	388	388	383	383	378
Estonia	43	43	43	47	47	47	47	50
Finland	578	578	578	589	589	607	607	600
France	1,075	1,084	1,084	1,089	1,089	1,094	1,094	1,016
Germany	1,842	1,842	1,842	1,851	1,851	1,915	1,915	1,668
Greece	141	140	140	152	152	153	153	139
Hungary	229	229	229	239	239	245	245	237
Ireland	109	109	109	114	114	113	113	105
Italy	943	943	943	996	996	1,009	1,009	1,048
Latvia	92	93	93	101	101	93	93	86
Lithuania	93	93	93	99	99	101	101	111
Luxembourg		15	15	15	15	15	15	15
Malta							2	2
Netherlands	209	210	210	211	211	213	213	375
Poland		817	817	817	817	869	869	858
Portugal	243	243	243	254	254	260	260	213
Romania							244	252
Slovak Republic	175	175	175	173	173	169	169	179
Slovenia	98	98	98	98	98	98	98	91
Spain	800	800	800	944	944	1,052	1,052	1,039
Sweden	705	705	705	730	730	755	755	770
UK	768	769	769	774	774	1,057	1,057	952
Σ	**9,420**	**10,282**	**10,282**	**10,605**	**10,605**	**11,186**	**11,432**	**11,116**
Bulgaria								128
Liechtenstein								2
Norway								113

Table 2.19: Number of installations covered by the EU ETS according to the different press releases. Source: European Union (2006, 2007, 2008a, 2009).

2.3c Trading volumes and permit prices of EUA futures

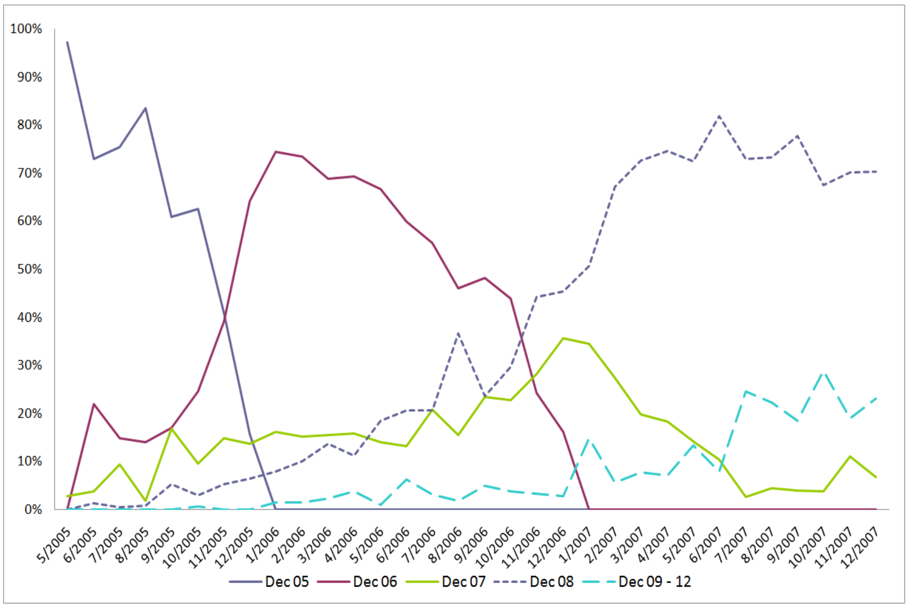

Figure 2.4: Breakdown of EUA futures trading volume by maturity of futures contract in Phase I (Monthly volume on the ECX in the period of May 2005 - December 2007).

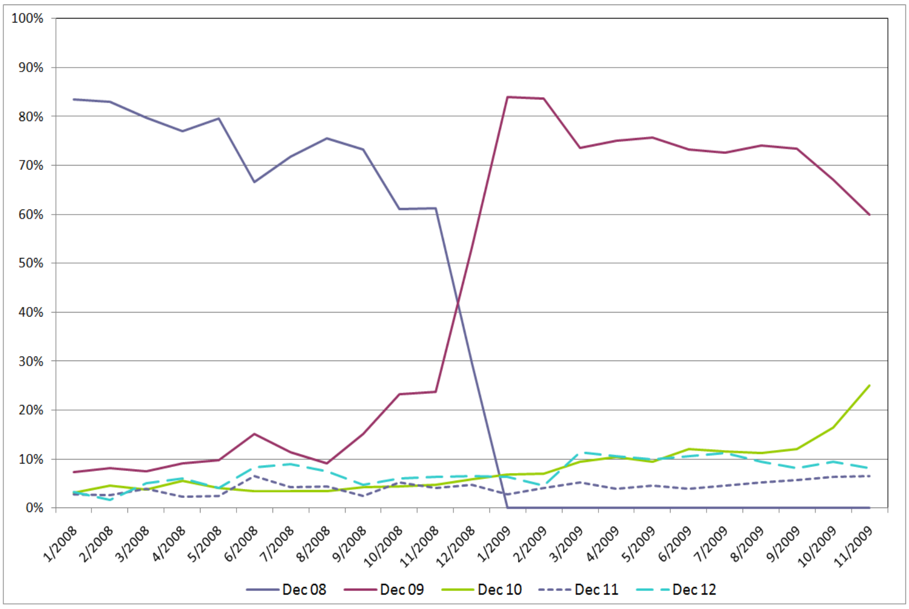

Figure 2.5: Breakdown of EUA futures trading volume by maturity of futures contract in Phase II (Monthly volume on the ECX in the period of January 2008 - November 2009).

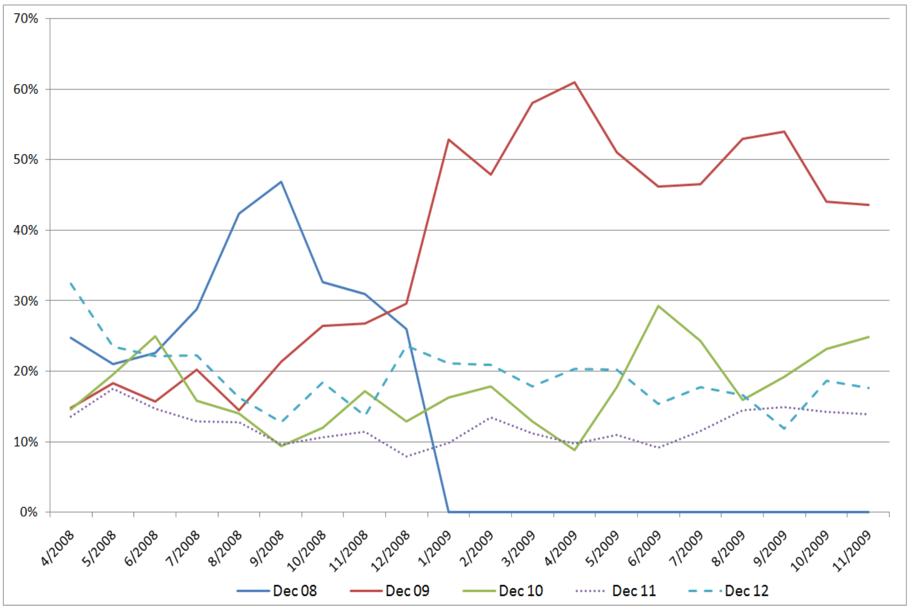

Figure 2.6: Breakdown of CER futures trading volume by maturity of futures contract in Phase II (Monthly volume on the ECX in the period of April 2008 - November 2009).

Figure 2.7: EUA futures prices between 22 April 2005 and 31 December 2007 (Phase I) as quoted on the ECX.

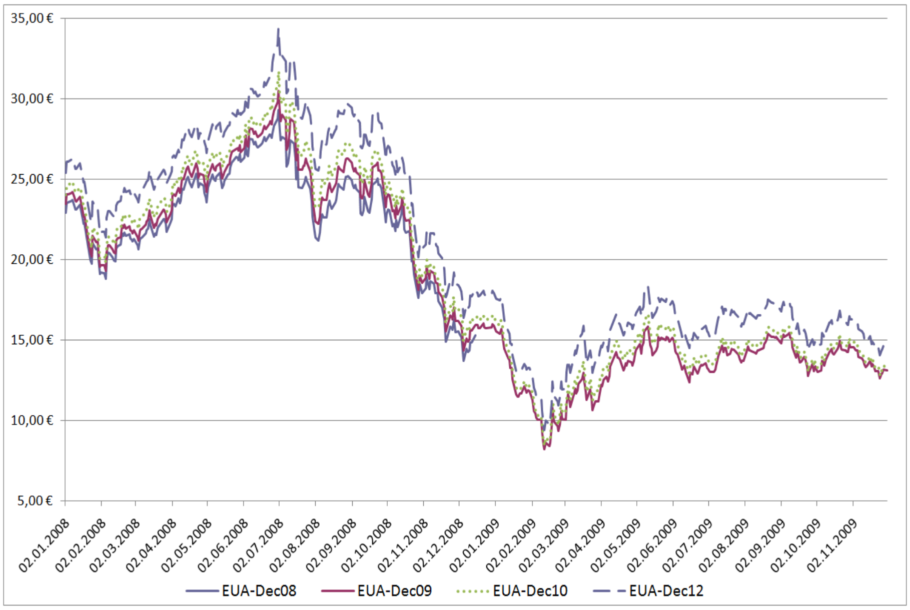

Figure 2.8: EUA futures prices between 2 January 2008 and 30 November 2009 (Phase II) as quoted on the ECX.

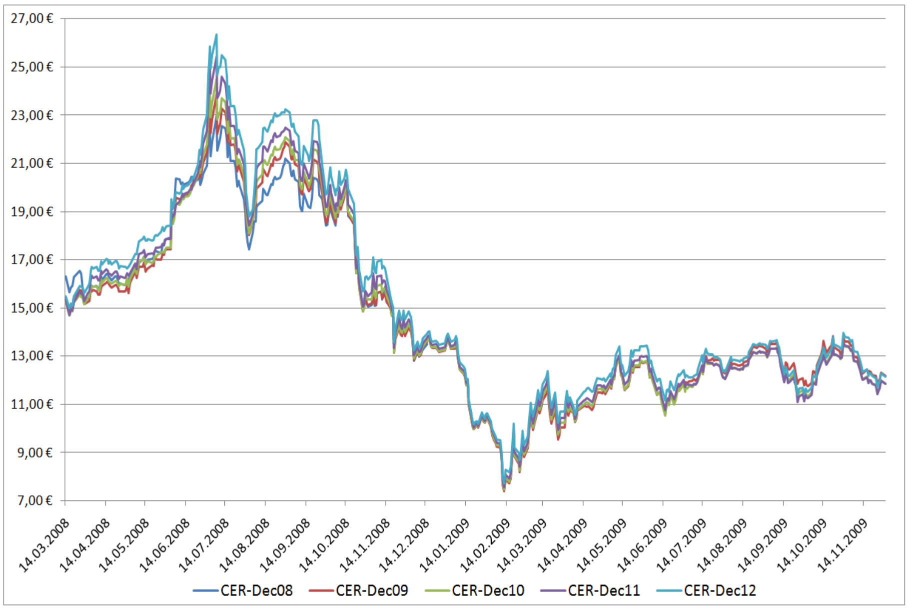

Figure 2.9: CER futures prices between 14 March 2008 and 30 November 2009 (Phase II) as quoted on the ECX.

Figure 2.10: EUA-Dec12 and CER-Dec12 futures prices between 14 March 2008 and 30 November 2009 (Phase II) as quoted on the ECX.

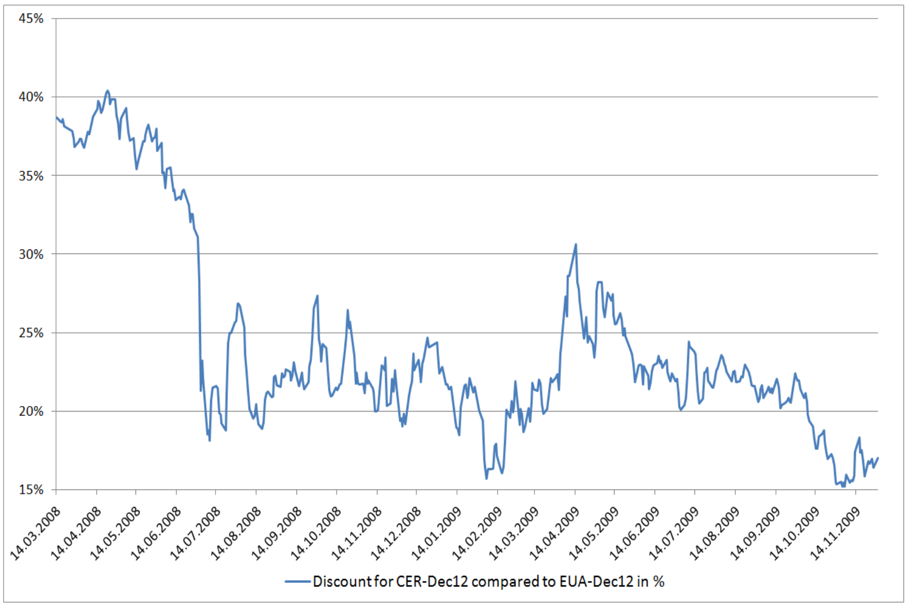

Figure 2.11: EUA-CER-spread between 14 March 2008 - 30 November 2009 (Phase II). Analysis based on ECX price quotes.

Chapter 3

Models

This chapter provides an overview of deterministic and stochastic equilibrium models and of reduced-form models. Furthermore, the relationship between these classes of models is analyzed. The notation of some models, as introduced in the original papers, has been modified in order to reduce difficulties in comparing the different models. A summary of the variable names in the deterministic and stochastic equilibrium models is to be found in Section 3.4.

3.1 Deterministic equilibrium models

In the deterministic equilibrium models of this section it is assumed that regulated companies are optimizing their profits/costs by choosing an optimal permit trading strategy and an optimal emission level. Regulated companies are obliged to comply during the modelled (compliance) period. The optimization problems of this section do not explicitly take a possible penalty payment into account.

3.1a Model of Montgomery (1972)

Introduction

Crocker (1966) and Dales (1968) proposed the introduction of marketable permits. However, Montgomery (1972) was the first to set up a model for permit prices. Within his framework it is shown that emissions trading leads to a cost-optimal solution (third desired characteristic of a policy instrument - cf. Chapter 1). This means that a central planner that jointly minimizes the costs of all polluters cannot reduce aggregate costs of all polluters compared to the scenario of emissions trading.

The proof of cost-optimality in the model of Montgomery (1972) requires relatively little knowledge on Operational Research. Proving cost-optimality in the more sophisticated models (cf. Section 3.1 and 3.2) is much more difficult and involves many technicalities. However, the basic idea of those proofs is similar to the one of Montgomery (1972). Cost-optimality of emissions trading can be shown in six steps.

1. Define the market equilibrium as the situation where regulated companies minimize their costs resulting from emitting and trading emission allowances subject to the condition that each company complies with its emission target. Furthermore, trading volumes must satisfy so-called market clearing conditions. Especially, a company can only sell allowances if another company is buying them.

2. Define the joint cost minimum as the situation where a central planner minimizes aggregate costs of all companies subject to the condition that aggregated emissions of all companies are less or equal than the total number of emission allowances.

3. Derive the conditions under which the market equilibrium exists.

4. Derive the conditions under which a joint cost minimum exists.

5. Show that a solution of the joint cost minimization problem satisfies the conditions of a market equilibrium.

6. Show that the regulated companies' emissions in a market equilibrium are a solution of the joint cost minimization problem.

In the following we present the simplified and slightly modified version of the model of Montgomery (1972).
The original model is simplified by neglecting the spatial dimension of the pollution problem. This is possible because we are interested in regulating greenhouse gas emissions which belong to the class of global pollutants. The pollution level of global pollutants is not influenced by the location of the pollution activity. Only in the case of local pollutants the location matters. The model of Montgomery (1972) is the only equilibrium model that takes the spatial dimension of the pollution problem into account (cf. Section 3.1 and 3.2).

Having simplified the original model, we slightly modify the resulting model such that it is comparable to the other equilibrium models given in Section 3.1 and 3.2. The original model assumes that the location-dependent pollution concentration results from multiplying the dispersion factor with the corresponding emission quantities. Reducing the analysis to one location and assuming that the dispersion factor is the same across all locations (this is the case for global pollutants such as CO_2) the pollution concentration is the product of the constant dispersion factor and the emission quantity. The constant dispersion factor allows us to formulate the optimization problem in terms of the emission quantity instead of the pollution concentration.

Definitions

The model of Montgomery (1972) assumes that there are n regulated companies. In the following the superscript i refers to company i. The time horizon of the model is finite. Especially, it is a time-discrete model with only one time step where all variables are modelled by deterministic functions. The model does not take discounting explicitly into account. However, as it is a one-step model this might be easily incorporated by multiplying the objective function with a discount factor. We use the following notation:

- Q^i denotes the emission quantity
- N^i is the number of emission allowances
- θ^i is the number of permits bought from or sold to other companies. Positive and negative values correspond to buying and selling, respectively.
- $\bar{S}$ denotes the permit price
- $C^i(Q^i)$ are the costs that firm i faces when adopting the emission level Q^i. It is defined as the difference between firm i's maximum profit when it is not obliged to achieve an emission target and its maximum profit when firm i must adopt the emission level Q^i.
  Denote the price of good r by G^r and assume that firm i produces $y^i = (y^{i,1}, \dots, y^{i,R})$ of the goods $1, \dots, R$. When adopting the emission level Q^i, production costs are equal to the twice differentiable and convex function $C^i_{\text{good}}(y^i, Q^i)$. Given y^i and Q^i, firm i's profit from producing goods are equal to

$$\pi^i(y^i, Q^i) = R^i_{\text{good}}(y^i) - C^i_{\text{good}}(y^i, Q^i) = \sum_{r=1}^{R} G^r y^{i,r} - C^i_{\text{good}}(y^i, Q^i).$$

Therefore, we can write $C^i(Q^i)$ as

$$C^i(Q^i) = \max_{y^i,Q^i} \left\{\pi^i(y^i,Q^i)\right\} - \max_{y^i} \left\{\pi^i(y^i,Q^i)\right\}.$$

Montgomery (1972) has shown that $C^i(Q^i)$ is a twice differentiable, decreasing and convex function.

Definition 3.1.1 (Market equilibrium)
Let $\bar{Q} = (\bar{Q}^1,\ldots,\bar{Q}^n) \geq 0$, $\bar{\theta} = (\bar{\theta}^1,\ldots,\bar{\theta}^n)$ *and* $S \geq 0$. *A market equilibrium is the (2n+1)-tuple* $(\bar{Q},\bar{\theta},\bar{S})$ *such that* $\bar{Q}$ *and* $\bar{\theta}$ *are the solution of the following n maximization problems:*

$$\begin{gathered}\max_{Q^i,\theta^i} \left\{-C^i(Q^i) - \bar{S}\theta^i\right\} \\ \textit{subject to} \quad N^i + \theta^i - Q^i \geq 0\end{gathered}$$

which also satisfy the market clearing conditions

$$\sum_{i=1}^{n} \bar{\theta}^i \leq 0 \quad \textit{and} \quad \bar{S}\sum_{i=1}^{n} \bar{\theta}^i = 0.$$

Remark:
(a) $N^i + \theta^i - Q^i \geq 0$ and $Q^i \geq 0$ imply $\theta^i \geq -N^i$.
(b) The maximization problems are equivalent to the following minimization problems:

$$\min_{Q^i,\theta^i} \left\{C^i(Q^i) + \bar{S}\theta^i\right\} \quad \text{subject to} \quad N^i + \theta^i - Q^i \geq 0.$$

The formulation as a maximization problem is chosen to be able to compare it to the other models in Section 3.1 and 3.2.

Definition 3.1.2 (Joint cost minimum)
A joint cost minimum is the n-tuple $\tilde{Q} = (\tilde{Q}^1,\ldots,\tilde{Q}^n) \geq 0$ *that is the solution of the following maximization problem:*

$$\begin{gathered}\max_{Q^1,\ldots,Q^n} \left\{-\sum_{i=1}^{n} C^i(Q^i)\right\} \\ \textit{subject to} \quad \sum_{i=1}^{n} \left(N^i - Q^i\right) \geq 0.\end{gathered} \tag{3.1}$$

Remark:
(a) The maximization problem can be formulated as the following minimization problem:

$$\min_{Q^1,\ldots,Q^n} \left\{\sum_{i=1}^{n} C^i(Q^i)\right\} \quad \text{subject to} \quad \sum_{i=1}^{n} \left(N^i - Q^i\right) \geq 0.$$

Solving static optimization problems

The maximization problems of the firms and the central planner are static linear optimization problems. This subsection provides an overview of the techniques that are necessary to solve these maximization problems.

Definition 3.1.3 (Lagrangian)
Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ *and let* $f(x), g_1(x), \dots, g_m(x)$ *be functions.*
Then the Lagrangian of the following static nonlinear optimization problem

$$\min_{x_1,\dots,x_n} \{f(x)\} \quad \textit{subject to} \quad g_j(x) \leq 0 \quad \textit{for } j = 1, \dots, m$$

is given by

$$L(x, u) = f(x) + \sum_{j=1}^{m} u_j g_j(x).$$

Definition 3.1.4 (Static convex optimization with non-negative control variables)
Let $x = (x_1, \dots, x_{n'}, x_{n'+1}, \dots, x_n)$ *. Assume that* $f(x), g_1(x), \dots, g_m(x)$ *are convex functions that are continuously differentiable. Furthermore, assume that there exists* $\tilde{x} \in \mathbb{R}^n$ *such that* $g_j(\tilde{x}) < 0$ *holds for all non-linear constraints. We consider the following optimization problem:*

$$\begin{aligned} &\min_{x_1,\dots,x_n} \{f(x)\} \\ &\textit{subject to} \quad g_j(x) \leq 0 \quad \textit{for } j = 1, \dots, m \textit{ and} \\ &\qquad\qquad\quad x_i \geq 0 \qquad \textit{for } i = 1, \dots, n' \ (n' \leq n). \end{aligned}$$

Theorem 3.1.5 (Karush-Kuhn-Tucker conditions)
$\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$ *is the optimal solution of the optimization problem of Definition 3.1.4 if and only if there exists* $\bar{u} \in \mathbb{R}^m$ *such that all the Karush-Kuhn-Tucker conditions are satisfied:*
For $i = 1, \dots, n'$

$$\frac{\partial L}{\partial x_i}(\bar{x}, \bar{u}) = \frac{\partial f}{\partial x_i}(\bar{x}) + \sum_{j=1}^{m} \bar{u}_j \frac{\partial g_j}{\partial x_i}(\bar{x}) \geq 0, \tag{3.2}$$

$$\bar{x}_i \frac{\partial L}{\partial x_i}(\bar{x}, \bar{u}) = \bar{x}_i \left[\frac{\partial f}{\partial x_i}(\bar{x}) + \sum_{j=1}^{m} \bar{u}_j \frac{\partial g_j}{\partial x_i}(\bar{x}) \right] = 0, \tag{3.3}$$

$$\bar{x}_i \geq 0, \tag{3.4}$$

and for $i = n' + 1, \ldots, n$

$$\frac{\partial L}{\partial x_i}(\bar{x}, \bar{u}) = \frac{\partial f}{\partial x_i}(\bar{x}) + \sum_{j=1}^{m} \bar{u}_j \frac{\partial g_j}{\partial x_i}(\bar{x}) = 0, \tag{3.5}$$

and for $j = 1, \ldots, m$

$$\frac{\partial L}{\partial u_j}(\bar{x}, \bar{u}) = g_j(\bar{x}) \leq 0, \tag{3.6}$$

$$\bar{u}_j \frac{\partial L}{\partial u_j}(\bar{x}, \bar{u}) = \bar{u}_j g_j(\bar{x}) = 0, \tag{3.7}$$

$$\bar{u}_j \geq 0. \tag{3.8}$$

Remark:
Depending on the book (cf. Neumann and Morlock (1993) and Hillier and Liebermann (1986)), the Karush-Kuhn-Tucker conditions are either derived for the following minimization problem with convex f and g

$$\min_x \{f(x)\} \quad \text{subject to} \quad g(x) \leq 0, \quad x \geq 0$$

or for the following maximization problem with concave $\tilde{f}$ and $\tilde{g}$

$$\max_x \left\{\tilde{f}(x)\right\} \quad \text{subject to} \quad \tilde{g}(x) \geq 0, \quad x \geq 0.$$

Setting $f = -\tilde{f}$ and $g = -\tilde{g}$ shows that both formulations are equivalent. Therefore, the Karush-Kuhn-Tucker conditions are the same for both optimization problems.
The following two optimization problems are also equivalent - the minimization problem with convex f and concave g

$$\min_x \{f(x)\} \quad \text{subject to} \quad g(x) \geq 0, \quad x \geq 0$$

and the maximization problem with concave f and convex g

$$\max_x \{f(x)\} \quad \text{subject to} \quad g(x) \leq 0, \quad x \geq 0.$$

Table 3.1 lists the Karush-Kuhn-Tucker conditions in those four cases. Most conditions remain the same as in Theorem 3.1.5. Only Equation (3.2) and (3.8) are varying.

	Constraint	
	Convex $g \leq 0$	Concave $g \geq 0$
maximization	$\frac{\partial L}{\partial x} \leq 0$	$\frac{\partial L}{\partial x} \geq 0$
of concave f	$u \leq 0$	$u \geq 0$
minimization	$\frac{\partial L}{\partial x} \geq 0$	$\frac{\partial L}{\partial x} \leq 0$
of convex f	$u \geq 0$	$u \leq 0$

Table 3.1: Overview of Karush-Kuhn-Tucker conditions for convex or concave target function and constraints.

Existence of market equilibrium and joint cost minimum

Theorem 3.1.6 (Existence of market equilibrium)
A market equilibrium $(\bar{Q}, \bar{\theta}, \bar{S})$ *exists if and only if there exist* $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_n) \geq 0$ *and* $\bar{S} \geq 0$ *such that* $(\bar{u}, \bar{S})$ *satisfy the Karush-Kuhn-Tucker conditions of the n maximization problems*

$$-\frac{\partial C^i}{\partial Q^i}(\bar{Q}^i) - \bar{u}_i \geq 0, \quad \bar{Q}^i \left[\tfrac{\partial C^i}{\partial Q^i}(\bar{Q}^i) + \bar{u}_i\right] = 0, \quad \bar{Q}^i \geq 0, \tag{3.9}$$

$$-\bar{S} + \bar{u}_i = 0, \tag{3.10}$$

$$N^i + \bar{\theta}^i - \bar{Q}^i \geq 0, \quad \bar{u}_i \left[N^i + \bar{\theta}^i - \bar{Q}^i\right] = 0, \quad \bar{u}_i \geq 0, \tag{3.11}$$

and the market clearing conditions

$$\sum_{i=1}^{n} \bar{\theta}^i \leq 0, \quad \bar{S} \textstyle\sum_{i=1}^{n} \bar{\theta}^i = 0. \tag{3.12}$$

Proof :
$C^i(Q^i)$ is a twice differentiable and convex function in Q^i . $\bar{S}\theta^i$ and $N^i+\theta^i-Q^i$ are linear functions. It follows that both $-C^i(Q^i) - \bar{S}\theta^i$ and $N^i + \theta^i - Q^i$ are twice differentiable and concave functions. Theorem 3.1.5 completes the proof. $\diamondsuit$

Remark:
Theorem 3.1.6 shows that in the market equilibrium the permit price $\bar{S}$ is equal to the marginal abatement costs $-\frac{\partial C^i}{\partial Q^i}(\bar{Q}^i)$ if firms are emitting ($\bar{Q}^i > 0$). This follows directly from Equation (3.9) and (3.10).

Theorem 3.1.7 (Existence of joint cost minimum)
A joint cost minimum $\tilde{Q}$ exists if and only if there exists $\tilde{u} \geq 0$ satisfying the Karush-Kuhn-Tucker conditions of the maximization problem given in Equation (3.1):

$$-\frac{\partial C^i}{\partial Q^i}(\tilde{Q}^i) - \tilde{u} \geq 0, \quad \sum_{i=1}^n \tilde{Q}^i \left[\frac{\partial C^i}{\partial Q^i}(\tilde{Q}^i) + \tilde{u}\right] = 0, \quad \bar{Q}^i \geq 0, \tag{3.13}$$

$$\sum_{i=1}^n (N^i - \tilde{Q}^i) \geq 0, \qquad \tilde{u}\left[\sum_{i=1}^n (N^i - \tilde{Q}^i)\right] = 0, \qquad \tilde{u} \geq 0. \tag{3.14}$$

Proof :
$C^i(Q^i)$ is a twice differentiable and convex function in Q^i implies that $-\sum_{i=1}^n C^i(Q^i)$ is twice differentiable and concave. Furthermore, we have a linear constraint. Therefore, Theorem 3.1.5 completes the proof. $\diamondsuit$

Relationship between optimality conditions

Lemma 3.1.8
A solution of the joint cost minimization problem satisfies the conditions of a market equilibrium.

Proof :
Using the conditions given in (3.13) and (3.14) we show that

$$\bar{Q}^i = \tilde{Q}^i, \qquad N^i + \bar{\theta}^i - \tilde{Q}^i = 0, \qquad \bar{u}_i = \tilde{u} = \bar{S}$$

satisfy the conditions given in (3.9) - (3.12).
Conditions (3.13) and (3.14) imply Equation (3.9):
Since $\frac{\partial C^i}{\partial Q^i}(\tilde{Q}^i) + \tilde{u} \leq 0$ and $\tilde{Q}^i \geq 0$ for all $i = 1, \ldots, n$, it follows from $\sum_{i=1}^n \tilde{Q}^i \left[\frac{\partial C^i}{\partial Q^i}(\tilde{Q}^i) + \tilde{u}\right] = 0$ that $\tilde{Q}^i \left[\frac{\partial C^i}{\partial Q^i}(\tilde{Q}^i) + \tilde{u}\right] = 0$ holds for all $i = 1, \ldots, n$. Therefore $\tilde{Q}^i$ and $\tilde{u}$ satisfy Equation (3.9) for all $i = 1, \ldots, n$.
Conditions (3.13) and (3.14) imply Equation (3.10):
If $\bar{u}_i = \tilde{u} = \bar{S}$, $\bar{S} - \bar{u}_i = 0$ is satisfied for all $i = 1, \ldots, n$ by any $\bar{\theta}^i$.
Conditions (3.13) and (3.14) imply Equation (3.11):
By $\bar{Q}^i = \tilde{Q}^i$ and $N^i + \bar{\theta}^i - \tilde{Q}^i = 0$, Equation (3.11) is satisfied for any $\bar{u}_i$.
Conditions (3.13) and (3.14) imply Equation (3.12):

$$0 \overset{(3.14)}{\leq} \sum_{i=1}^n (N^i - \tilde{Q}^i) \overset{N^i + \bar{\theta}^i - \tilde{Q}^i = 0}{=} -\sum_{i=1}^n \bar{\theta}^i,$$

$$0 \overset{(3.14)}{=} \tilde{u}\left[\sum_{i=1}^{n}(N^i - \tilde{Q}^i)\right] \overset{\bar{S}=\tilde{u}}{=} \bar{S}\left[\sum_{i=1}^{n}(N^i - \tilde{Q}^i)\right] \overset{N^i+\bar{\theta}^i-\tilde{Q}^i=0}{=} -\bar{S}\sum_{i=1}^{n}\bar{\theta}^i.$$

$\diamondsuit$

Lemma 3.1.9
Any emission vector that satisfies the conditions of a market equilibrium is a solution of the joint cost miminization problem.

Proof :
Using the conditions given in (3.9) - (3.12) we show that

$$\tilde{Q}^i = \bar{Q}^i, \qquad \tilde{u} = \bar{S}$$

satisfy the conditions given in (3.13) and (3.14).
Conditions (3.9) - (3.12) imply Equation (3.13):
By Equation (3.10), $\bar{u}_i = \bar{S}$. Therefore,

$$-\frac{\partial C^i}{\partial Q^i}(\bar{Q}^i) - \bar{S} \geq 0, \quad \bar{Q}^i\left[\tfrac{\partial C^i}{\partial Q^i}(\bar{Q}^i) + \bar{S}\right] = 0,$$

which implies $\sum_{i=1}^{n}\bar{Q}^i\left[\frac{\partial C^i}{\partial Q^i}(\bar{Q}^i) + \bar{S}\right] = 0$. Therefore, $\bar{Q}^i$ and $\tilde{u} = \bar{S}$ satisfy Equation (3.13).
Conditions (3.9) - (3.12) imply Equation (3.14):
By Equation (3.11) and (3.12), $\sum_{i=1}^{n}(N^i - \bar{Q}^i) \geq -\sum_{i=1}^{n}\bar{\theta}^i \geq 0$.
By Equation (3.10), Equation (3.11) becomes $\bar{S}\left[N^i + \bar{\theta}^i - \bar{Q}^i\right] = 0$. By Equation (3.12),

$$0 = \sum_{i=1}^{n}\bar{S}\left[N^i + \bar{\theta}^i - \bar{Q}^i\right] = \bar{S}\sum_{i=1}^{n}(N^i - \bar{Q}^i) + \bar{S}\sum_{i=1}^{n}\bar{\theta}^i = \bar{S}\sum_{i=1}^{n}(N^i - \bar{Q}^i).$$

Therefore, $\bar{Q}^i$ and $\tilde{u} = \bar{S}$ satisfy Equation (3.14). $\diamondsuit$

Results

In the model of Montgomery (1972)

- the permit price equals marginal abatement costs
- the market equilibrium is equivalent to the joint cost minimum

3.1b Model of Rubin (1996)

Introduction

Rubin (1996) extends the work of Montgomery (1972) by proving cost-optimality of emissions trading in a continuous-time framework. This setting allows the analysis of the permit price evolution over time. Such an analyis is not possible in the discrete-time model of Montgomery (1972) with one time-step. Apart from showing the cost-optimality of emissions trading, the paper of Rubin (1996) is concerned with the effect of banking and borrowing on the permit price. However, the following subsection focuses on the main part of the proof of cost-optimality of emissions trading.

Definitions

The model assumes that there are n regulated companies. In the following the superscript i refers to company i. The time horizon of the model is finite. Especially, it is a time-continuous model in the interval $[0, T]$ where all variables are modelled by deterministic functions.

We use the following notation - similar to the one used in Section 3.1a:

- Q_t^i denotes the emission quantity at time t
- N_t^i is the number of emission allowances that is allocated at time t
- θ_t^i is the number of permits bought from or sold to other companies at time t. Positive and negative values correspond to buying and selling, respectively.
- $\bar{S}_t$ denotes the permit price at time t
- $C^i(Q_t^i)$ are the costs that firm i faces when adopting the emission level Q_t^i. The twice differentiable, decreasing and convex function $C^i(\cdot)$ is defined like in the model of Montgomery (1972).
- B_t^i models the number of permits that are available after the compliance commitment at time t. Positive/Negative values for B_t^i mean that firm i is banking/borrowing permits at time t.

Definition 3.1.10 (Market equilibrium)
A market equilibrium consists of the vectors $\bar{Q}_t = (\bar{Q}_t^1, \ldots, \bar{Q}_t^n) \geq 0$, $\bar{\theta}_t = (\bar{\theta}_t^1, \ldots, \bar{\theta}_t^n)$ *and* $\bar{S}_t \geq 0$ *such that* $\bar{Q}_t$ *and* $\bar{\theta}_t$ *maximize the following optimization problem of each firm given* $\bar{S}_t$ *:*

$$\max_{Q_t^i, \theta_t^i} \left\{ \int_0^T e^{-rt} \left[-C^i(Q_t^i) - \bar{S}_t \theta_t^i \right] \right\}$$

subject to the constraints of each firm for each $t \in [0, T]$

$$\begin{aligned} \dot{B}_t^i &:= \frac{\partial B_t^i}{\partial t} = N_t^i + \theta_t^i - Q_t^i, \\ B_0^i &= 0, \quad B_t^i \geq 0, \\ Q_t^i &\geq 0, \end{aligned}$$

which also satisfy the market clearing condition on permits and the terminal stock condition

$$\begin{aligned} \sum_{i=1}^n \bar{\theta}_t^i &= 0, \\ \bar{S}_T \sum_{i=1}^n \bar{B}_T^i &= 0. \end{aligned}$$

Remark:
(a) Furthermore, the following technical constraint must hold

$$-\lambda_t^i \leq \theta_t^i \leq \Lambda_t^i, \qquad \lambda_t^i > 0, \qquad \Lambda_t^i > 0. \tag{3.15}$$

(b) As in the model of Montgomery (1972), the abatement-cost function $C^i(Q_t^i)$ is a twice differentiable, decreasing and convex function that models the difference between the unconstrained profits and the profits when the regulated company is adopting the emission level Q_t^i.
(c) The constraint $B_t^i \geq 0$ for all $t \in [0, T]$ means that borrowing is always forbidden and that banking is allowed if $B_t^i > 0$.

Definition 3.1.11 (Joint cost minimum)
A joint cost minimum $\tilde{Q}_t = (\tilde{Q}_t^1, \ldots, \tilde{Q}_t^n) \geq 0$ *is the solution of the following maximization problem:*

$$\max_{Q_t^1, \ldots, Q_t^n} \left\{ \int_0^T e^{-rt} \left[-\sum_{i=1}^n C^i(Q_t^i) \right] \right\}$$

subject to the constraints of each firm for each $t \in [0, T]$

$$\begin{aligned}
\dot{B}_t &:= \frac{\partial B_t}{\partial t} = \sum_{i=1}^{n} (N_t^i - Q_t^i), \\
B_0 &= 0, \quad B_t \geq 0, \\
Q_t^i &\geq 0 \textit{ for all } i = 1, \ldots, n.
\end{aligned}$$

Solving dynamic optimization problems

Theorem 3.1.12 (Dynamic optimization problem)
Let $T < \infty$ *and let* f *and* g *be twice differentiable concave functions. Consider the following dynamic, deterministic optimization problem*

$$\max_{x(t)} \left\{ \int_0^T f(s(t), x(t), t) dt \right\} \tag{3.16}$$

$$\textit{subject to} \quad \dot{s}(t) := \frac{\partial s}{\partial t}(t) = g(s(t), x(t), t) \textit{ for } t \in [0, T] \textit{ and} \tag{3.17}$$

$$s(0) = 0 \textit{ and} \tag{3.18}$$

$$s(T) \geq 0. \tag{3.19}$$

Then the Hamiltonian is defined as

$$H := H(s(t), x(t), t) = f(s(t), x(t), t) + u(t) \cdot g(s(t), x(t), t) \tag{3.20}$$

and the solution of the optimization problem must satisfy

$$\frac{\partial H}{\partial x} = 0, \tag{3.21}$$

$$-\frac{\partial H}{\partial s} = \frac{\partial u}{\partial t} =: \dot{u}, \tag{3.22}$$

$$-\frac{\partial H}{\partial u} = \frac{\partial s}{\partial t} =: \dot{s}, \tag{3.23}$$

$$u(T)s(T) = 0. \tag{3.24}$$

Remark:

(a) $x(t)$ is called control variable. The state variable $s(t)$ is influenced by the choice of the control variable.However, the planner is not able to control it directly.

(b) Equation (3.21) is similar to the condition in a static non-linear optimization problem.

(c) Equation (3.23) restates the condition on the state variable (cf. Equation (3.17)).

(d) Equation (3.24) is called transversality condition.

Proof (Idea):
A rigorous proof can be found in Pontryagin et al. (1962). The following proof is along the lines of Barro and Sala-i-Martin (1995).

First, we rewrite the constraint as an integral and set up the Lagrangian function with the continuum of multipliers $u(t)$ for the dynamic constraint and the multiplier v for the terminal condition of the state variable:

$$L = \int_0^T f(s(t), x(t), t)dt + \int_0^T u(t) \cdot [g(s(t), x(t), t) - \dot{s}(t)]\, dt + vs(T).$$

Second, integration by parts of

$$\begin{aligned}\int_0^T u(t)\dot{s}(t)dt &= [u(t)s(t)]_0^T - \int_0^T \dot{u}(t)s(t)dt = u(T)s(T) - u(0)s(0) - \int_0^T \dot{u}(t)s(t)dt \\ &= u(T)s(T) - \int_0^T \dot{u}(t)s(t)dt\end{aligned}$$

and using the definition of the Hamiltonian yields

$$L = \int_0^T [H(s(t), x(t), t) + \dot{u}(t)s(t)]\, dt + (v - u(T)) \cdot s(T). \tag{3.25}$$

Third, solve the problem using perturbation analysis:
Let $\bar{x}(t)$ be the optimal path for the control variable. The constraint $\dot{s}(t) = g(s(t), x(t), t)$ yields an optimal path for the state variable that we denote by $\bar{s}(t)$. Define the perturbations of the optimal paths by

$$x := x(t) = \bar{x}(t) + \varepsilon p^{(x)}(t), \quad s := s(t) = \bar{s}(t) + \varepsilon p^{(s)}(t), \quad s(T) = \bar{s}(T) + \varepsilon dS(T),$$

where ε is a scalar and $p^{(x)} := p^{(x)}(t)$ and $p^{(s)} := p^{(s)}(t)$ are called perturbation functions. The perturbation analysis is completed by using that near the optimum small perturbations do not affect the maximum value of our optimization problem, i.e.

$$\frac{\partial L}{\partial \varepsilon}(\bar{s}(t), \bar{x}(t), t) = 0. \tag{3.26}$$

Applying the chain rule to Equation (3.25) yields

$$\begin{aligned}\frac{\partial L}{\partial \varepsilon} &= \int_0^T \left[\frac{\partial H}{\partial s} \cdot \frac{\partial s}{\partial \varepsilon} + \frac{\partial H}{\partial x} \cdot \frac{\partial x}{\partial \varepsilon} + \dot{u} \cdot \frac{\partial s}{\partial \varepsilon}\right] dt + (v - u(T)) \cdot \frac{\partial s(T)}{\partial \varepsilon} \\ &= \int_0^T \left[\frac{\partial H}{\partial s} \cdot p^{(s)} + \frac{\partial H}{\partial x} \cdot p^{(x)} + \dot{u} \cdot p^{(s)}\right] dt + (v - u(T)) \cdot dS(T)\end{aligned}$$

$$= \int_0^T \left[\frac{\partial H}{\partial x} \cdot p^{(x)} + \left(\frac{\partial H}{\partial s} + \dot{u} \right) \cdot p^{(s)} \right] dt + (v - u(T)) \cdot dS(T)$$

Since $\frac{\partial L}{\partial \varepsilon}(\bar{s}(t), \bar{x}(t), t) = 0$ must hold for any choice of perturbation functions, we obtain

$$\frac{\partial H}{\partial x} = 0, \quad \frac{\partial H}{\partial s} + \dot{u} = 0, \quad v = u(T).$$

Combining $v = u(T)$ and $v \cdot s(T) = 0$, the complementary slackness condition from the terminal constraint, yields the so-called transversality condition

$$u(T)s(T) = 0.$$

$\diamondsuit$

Existence of market equilibrium and joint cost minimum

Theorem 3.1.13 (Existence of market equilibrium)
Under the assumption that regulated companies are not buying or selling at the minimum or maximum rate (cf. Equation (3.15)), a market equilibrium $(\bar{Q}_t, \bar{\theta}_t, \bar{S}_t)$ *exists if and only if there exist for all* $t \in [0, T]$ *non-negative multipliers* $\bar{u}_t = (\bar{u}_t^1, \ldots, \bar{u}_t^n), \bar{\beta}_t = (\bar{\beta}_t^1, \ldots, \bar{\beta}_t^n)$ *and an optimal permit price* $\bar{S} \geq 0$ *such that the following conditions hold for all* $i = 1, \ldots, n$ *:*

$$e^{-rt} \frac{\partial C^i}{\partial Q_t^i}(\bar{Q}_t^i) - \bar{u}_t^i \geq 0, \quad \tfrac{\partial \bar{u}_t^i}{\partial t} = \bar{\beta}_t, \quad e^{-rt} \bar{S}_t + \bar{u}_t^i = 0, \tag{3.27}$$

$$\bar{Q}_t^i \left[e^{-rt} \frac{\partial C^i}{\partial Q_t^i}(\bar{Q}_t^i) - \bar{u}_t^i \right] = 0, \quad \bar{B}_t^i \bar{\beta}_t^i = 0, \quad \bar{B}_T^i \bar{u}_T^i = 0, \tag{3.28}$$

$$\bar{Q}_t^i \geq 0, \quad \bar{B}_t^i \geq 0, \quad \frac{\partial B_t^i}{\partial t} = N_t^i + \bar{\theta}_t^i - \bar{Q}_t^i, \tag{3.29}$$

and the market clearing condition and the terminal stock condition

$$\sum_{i=1}^n \bar{\theta}_t^i = 0, \quad \bar{S}_T \sum_{i=1}^n \bar{B}_T^i = 0. \tag{3.30}$$

Proof (Idea):
Forming the Langrangian of the corresponding minimization problem

$$L^i = e^{-rt} \left(C^i(Q_t^i) + \bar{S}_t \theta_t^i \right) + u_t^i (N_t^i + \theta_t^i - Q_t^i) - \beta_t^i B_t^i$$

and using both the conditions given in Equation (3.21) - (3.24) (cf. Theorem 3.1.12) and the Karush-Kuhn-Tucker conditions completes the proof. The conditions can then be retrieved from

$$\frac{\partial L^i}{\partial Q_t^i} \geq 0, \quad \tfrac{\partial \bar{u}_t^i}{\partial t} = -\tfrac{\partial L^i}{\partial B_t^i}, \quad \frac{\partial L^i}{\partial \theta_t^i} = 0,$$

$$\bar{Q}_t^i \frac{\partial L^i}{\partial Q_t^i} = 0, \quad \bar{B}_t^i \tfrac{\partial L^i}{\partial B_t^i} = 0, \quad \bar{B}_T^i \bar{u}_T^i = 0,$$
$$\bar{Q}_t^i \geq 0, \quad \bar{B}_t^i \geq 0, \quad \frac{\partial L^i}{\partial u_t^i} = \frac{\partial B_t^i}{\partial t}.$$

$\diamondsuit$

Interpretation

(a) If $\bar{Q}_t^i > 0$, then

$$\bar{u}_t^i = e^{-rt} \frac{\partial C^i}{\partial Q_t^i}(\bar{Q}_t^i), \qquad \bar{S}_t = -\frac{\partial C^i}{\partial Q_t^i}(\bar{Q}_t^i).$$

This means that the permit price is equal to the marginal abatement costs. Moreover, the marginal value of a banked emission permit is equal to the discounted marginal abatement costs.

(b) Rubin (1996) shows that the permit price is growing at the risk-free interest rate if there are no restrictions on banking and borrowing. However, the permit price will have a lower growth rate in case borrowing is forbidden but regulated companies wish to do so.

(c) $\bar{B}_T^i \bar{u}_T^i = 0$, also known as transversality condition, implies that if a company still holds permits in the bank at time T, the value of those permits is zero.

Theorem 3.1.14 (Existence of joint cost minimum)
A joint cost minimum exists if and only if there exist for all $t \in [0,T]$ non-negative multipliers $\tilde{u}_t, \tilde{\beta}_t$ such that the following conditions hold for $i = 1, \dots, n$:

$$e^{-rt} \frac{\partial C^i}{\partial Q_t^i}(\tilde{Q}_t^i) - \tilde{u}_t \geq 0, \quad \tfrac{\partial \tilde{u}_t}{\partial t} = \tilde{\beta}_t, \quad \frac{\partial B_t}{\partial t} = \sum_{i=1}^{n} (N_t^i - \tilde{Q}_t^i), \tag{3.31}$$

$$\tilde{Q}_t^i \left[e^{-rt} \frac{\partial C^i}{\partial Q_t^i}(\tilde{Q}_t^i) - \tilde{u}_t \right] = 0, \quad \tilde{B}_t \tilde{\beta}_t = 0, \quad \tilde{B}_T \tilde{u}_T = 0, \tag{3.32}$$

$$\tilde{Q}_t^i \geq 0, \quad \tilde{B}_t \geq 0. \tag{3.33}$$

Proof :
Similar to Theorem 3.1.13. The conditions can be retrieved from

$$\frac{\partial L}{\partial Q_t^i} \geq 0, \quad \tfrac{\partial \tilde{u}_t}{\partial t} = -\tfrac{\partial L}{\partial B_t}, \quad \frac{\partial L}{\partial u_t} = \frac{\partial B_t}{\partial t},$$
$$\tilde{Q}_t^i \frac{\partial L}{\partial Q_t^i} = 0, \quad \tilde{B}_t \tfrac{\partial L}{\partial B_t} = 0, \quad \tilde{B}_T \tilde{u}_T = 0,$$

$$\tilde{Q}_t^i \geq 0, \qquad \tilde{B}_t \geq 0,$$

where the Lagrangian is given by

$$L = e^{-rt} \sum_{i=1}^{n} C^i(Q_t^i) + u_t \sum_{i=1}^{n} (N_t^i - Q_t^i) - \beta_t B_t.$$

$\diamondsuit$

Interpretation

(a) If $\tilde{Q}_t^i > 0$, then

$$-\tilde{u}_t = -e^{-rt} \frac{\partial C^1}{\partial Q_t^1}(\tilde{Q}_t^1) = \ldots = -e^{-rt} \frac{\partial C^n}{\partial Q_t^n}(\tilde{Q}_t^n). \tag{3.34}$$

This means that the marginal cost of banking permits is equal to the discounted marginal abatement costs. Especially, marginal abatement costs are the same for all the regulated companies that are emitting.

(b) $\tilde{B}_t > 0$ implies $\frac{\partial \tilde{u}_t}{\partial t} = 0$. In other words, if marginal costs of banking are not constant, banks will not bank.

Lemma 3.1.15

(a) A solution of the joint cost minimization problem satisfies the conditions of a market equilibrium.

(b) A solution that satisfies the conditions of a market equilibrium is a solution of the joint cost minimization problem.

Proof :
Follows from Theorem 3.1.13 and 3.1.14. See Rubin (1996). $\diamondsuit$

Results

In the model of Rubin (1996)

- the permit price equals marginal abatement costs
- the market equilibrium is equivalent to the joint cost optimum

3.1c Model of Kling and Rubin (1997)

Introduction

Kling and Rubin (1997) extend the work of Rubin (1996). The aim of their work is to highlight that it has to be distinguished between cost optimality and social optimality. Montgomery (1972) and Rubin (1996) show that emissions trading leads to a least-cost solution. Kling and Rubin (1997) include a so-called damage function into the optimization problem of the central planner and analyze whether emissions trading is also socially optimal.
The following subsection presents the most important definitions and results in the model of Kling and Rubin (1997). Mathematical derivations are not included as the basic idea is similar to the proof in Section 3.1b.

Definitions

The main difference between the model of Kling and Rubin (1997) and the model of Rubin (1996) is the introduction of a damage function in the central planner's maximization problem (cf. Definition 3.1.17). The convex damage function

$$D(Q, t)$$

describes the costs that the emission quantity Q imposes on the society.
The formulation of the maximization problem for an individual profit maximizing firm is also different in the framework of Kling and Rubin (1997):

- In addition to the costs from producing and abating, Kling and Rubin (1997) include the revenues from producing a good into the optimization problem. It is assumed that the firm i produces y_t^i quantities of output and that the time-t price for one unit of the good is denoted by G_t. Instantaneous revenues from the production of the good at time are given by

$$R^i_{\text{good}}(y^i_t) = G_t y^i_t.$$

- Kling and Rubin (1997) model the costs of producing and adopting a given emission level instead of directly including the abatement cost function in the optimization problem like Rubin (1996). The cost function $C^i_{\text{good}}(Q, y)$ describes the total costs of producing the output quantity y when adopting the emission level Q. It is assumed that $C^i_{\text{good}}(Q,y)$ is strictly convex in (Q,y), i.e.

$$\frac{\partial C^i_{\text{good}}}{\partial y} > 0, \quad \frac{\partial C^i_{\text{good}}}{\partial Q} < 0, \frac{\partial^2 C^i_{\text{good}}}{\partial y \partial Q} < 0, \quad \frac{\partial^2 C^i_{\text{good}}}{\partial y^2} > 0, \quad \frac{\partial^2 C^i_{\text{good}}}{\partial Q^2} > 0.$$

Definition 3.1.16 (Firm i's profit maximization problem)
Given the permit price $\bar{S}_t$, firm i chooses an optimal emission level $\bar{Q}^i_t \geq 0$, buys or sells an optimal number of permits $\bar{\theta}^i_t$ and produces an optimal quantity of output $\bar{y}^i_t$:

$$\max_{Q^i_t, \theta^i_t, y^i_t} \left\{ \int_0^T e^{-rt} \left[R^i_{good}(y^i_t) - C^i_{good}(Q^i_t, y^i_t) - \bar{S}_t \theta^i_t \right] dt \right\}$$

subject to the constraints of each firm for each $t \in [0,T]$

$$\begin{aligned} \dot{B}^i_t &:= \frac{\partial B^i_t}{\partial t} = N^i_t + \theta^i_t - Q^i_t, \\ B^i_0 &= 0, \quad B^i_T \geq 0, \\ Q^i_t &\geq 0. \end{aligned}$$

Remark:

(a) Similar to the model of Rubin (1996), the following technical constraint must hold

$$-\lambda^i_t \leq \theta^i_t \leq \Lambda^i_t, \qquad \lambda^i_t > 0, \qquad \Lambda^i_t > 0.$$

(b) The target function is similar to the one used in the model of Carmona et al. (2009b). Only the penalty payment is missing in the target function of Definition 3.1.16.

Definition 3.1.17 (Central planner's optimization problem)
The central planner chooses optimal emission levels $\bar{Q}_t = (\bar{Q}^1_t, \ldots, \bar{Q}^n_t) \geq 0$ and output quantities $\bar{y}_t = (\bar{y}^1_t, \ldots, \bar{y}^n_t)$:

$$\max_{Q^1_t, \ldots, Q^n_t, y^1_t, \ldots, y^n_t} \left\{ \int_0^T e^{-rt} \left[\sum_{i=1}^n R^i_{good}(y^i_t) - \sum_{i=1}^n C^i_{good}(Q^i_t, y^i_t) - D\left(\sum_{i=1}^n Q^i_t, t \right) \right] \right\}$$

subject to the constraints of each firm for each $t \in [0,T]$

$$\begin{aligned} \dot{B}_t &:= \frac{\partial B_t}{\partial t} = \sum_{i=1}^n (N^i_t - Q^i_t), \\ B_0 &= 0, \quad B_T \geq 0, \\ Q^i_t &\geq 0 \quad \text{for all } i = 1, \ldots, n \end{aligned}$$

Remark:
The target function of the central planner's optimization problem are the firm's aggregated revenues minus costs. In addition, the damage resulting from emissions is subtracted. Therefore, the central planner's optimization problem in the framework of Kling and Rubin (1997) differs from the one in the framework of Rubin (1996) w.r.t. the definition of revenues and costs and the inclusion of the convex damage function $D(Q_t, t)$.

Results

Kling and Rubin (1997) show that

- The permit price is equal to the marginal abatement costs
$$\bar{S}_t = -\frac{\partial C^1}{\partial Q_t^1}(\bar{Q}_t^1, \bar{y}_t^1) = \ldots = -\frac{\partial C^n}{\partial Q_t^n}(\bar{Q}_t^n, \bar{y}_t^n).$$
- Emissions trading does not necessarily lead to a socially optimal solution (in the sense of Definition 3.1.17).
However, under the assumption that social damages are linear and stationary, a social optimum is achieved by the introduction of modified banking rules that penalize borrowing by the discount rate. Borrowing one unit of emissions means that the firm emits one unit using an emission permit for compliance that is earmarked for the use in the future.
Under the banking regulations of Rubin (1996) a firm has to hand in one permit per unit of emission independent of when emissions took place. This banking regulations are described by the constraint
$$\dot{B}_t := \frac{\partial B_t}{\partial t} = \sum_{i=1}^{n}(N_t^i - Q_t^i). \tag{3.35}$$
The proposed modified banking rules of Kling and Rubin (1997) oblige firms that borrow one unit of emission for a time period of length t to hand in $e^{rt}(> 1)$ permits at time t. The modified banking rules can be translated into the following constraint
$$\dot{B}_t := \frac{\partial B_t}{\partial t} = e^{-rt}\sum_{i=1}^{n}(N_t^i - Q_t^i). \tag{3.36}$$
The modified banking rules of Kling and Rubin (1997) differ from the original banking rules by the discount factor (cf. Equation (3.35) and (3.36)).

3.1d Model of Cronshaw and Kruse (1996)

Introduction

The model of Cronshaw and Kruse (1996) is the discrete-time version of the models of Rubin (1996) and Kling and Rubin (1997). A mathematical derivation of the cost optimality of emissions trading in the model of Cronshaw and Kruse (1996) is not provided in this subsection. Except for mathematical technicalities the idea of the proof is the same as in the model of Rubin (1996). The models of Cronshaw and Kruse (1996) and Rubin (1996) have in common that both are concerned with the cost-optimality of emissions trading whereas the model of Kling and Rubin (1997) rather addresses the issue of social optimality of emissions trading. However, when comparing the target functions of the optimization problems it can be seen that both the models of Cronshaw and Kruse (1996) and Kling and Rubin (1997) maximize profits from the production of goods whereas the model of Rubin (1996) focuses on abatement costs.

Definitions

The modelling assumptions of Cronshaw and Kruse (1996) are the following

- The model of Cronshaw and Kruse (1996) is the only equilibrium model that explicitly specifies input variables. It is assumed that input choices for production x^i and abatement choices z^i are transformed by the functions f^i_{Output} and $f^i_{\text{Emissions}}$ into the output quantity y^i and the emission level Q^i .
- Firms are maximizing their profits from the production of goods and from emissions trading by choosing optimal input quantities, producing an optimal number of goods y^i , choosing an optimal emission volume Q^i , buying and selling permits optimally (choosing θ^i optimally). Firms are subject to borrowing constraints and are obliged to comply with their emission target. This can be translated into an optimization problem with the following target function
$$\max_{\theta^i, x^i, z^i} \left\{ \sum_{t=1}^{T} e^{-rt} \left[R^i_{\text{good}}(y^i_t) - C^i_{\text{good}}(x^i_t, z^i_t) - S^i_t \theta^i_t \right] \right\}.$$
This optimization problem is too complex and therefore, Cronshaw and Kruse (1996) split it up into two parts that are solved sequentially.
In the first step, given the emission volume Q^i and the number of traded permits θ^i ,

the firm is choosing optimal production and abatement inputs (x^i , z^i) and optimal output quantities for the produced good y^i . This yields the following definition: $\Pi^i(Q^i, \theta^i)$ denotes the maximum profit of firm i from the production of goods given the emission level Q^i and the number of traded permits θ^i , i.e.

$$\Pi^i(Q^i, \theta^i) = \max_{x^i, z^i} \left\{ \sum_{t=1}^{T} e^{-rt} \left[R^i_{\text{good}}(y^i_t) - C^i_{\text{good}}(x^i_t, z^i_t) \right] \right\}$$

subject to the constraints

$$\begin{aligned} y^i_t &= f^i_{\text{Output}}(x^i_t) \quad \text{for all } i = 1, \ldots, n, \\ f^i_{\text{Emissions}}(x^i_t, z^i_t) &\leq Q^i_t \quad \text{for all } i = 1, \ldots, n. \end{aligned}$$

Definition 3.1.18 (Firm i's profit maximization problem)
Given the permit price S , firm i chooses an optimal emission level Q^i and buys or sells an optimal number of permits θ^i :

$$\max_{Q^i, \theta^i} \left\{ \Pi^i(Q^i, \theta^i) - \sum_{t=1}^{T} e^{-rt} S^i_t \theta^i_t \right\} \tag{3.37}$$

subject to the constraints of each firm for each $t = 1, \ldots, T$

$$\begin{aligned} B^i_{t+1} - B^i_t &= N^i_t + \theta^i_t - Q^i_t, \\ B^i_1 &= 0, \quad B^i_t \geq 0 \textit{ for all } t = 2, \ldots, T, \\ Q^i_t &\geq 0. \end{aligned}$$

Remark:
In the market equilibrium consisting of the permit price S and the associated optimal strategies $(\bar{\theta}^1, \ldots, \bar{\theta}^n)$ and $(\bar{Q}^1, \ldots, \bar{Q}^n)$ all the firms are maximizing their profits and the following market clearing condition holds

$$\sum_{t=1}^{T} \bar{\theta}^i_t = 0 \text{ for } t = 1, \ldots, T.$$

Definition 3.1.19 (Joint optimization problem)
The central planner chooses optimal emission levels $Q^1, \ldots, Q^n$:

$$\max_{Q^1, \ldots, Q^n} \left\{ \sum_{i=1}^{n} \Pi^i(Q^i) \right\}$$

subject to the constraints of each firm for each $t \in [0, T]$

$$\begin{aligned} B_{t+1} - B_t = & \sum_{i=1}^{n} (N_t^i - Q_t^i), \\ & B_t \geq 0 \textit{ for all } t = 1, \ldots, T, \\ & Q_t^i \geq 0. \end{aligned}$$

Results

- Cronshaw and Kruse (1996) show that the market equilibrium is equivalent to the joint cost optimum.
- The permit price equals marginal abatement costs.

3.2 Stochastic equilibrium models

3.2a Model of Seifert et al. (2008)

Introduction

The model of Seifert et al. (2008) is the first paper that addresses the issue of permit price dynamics. Two other papers have been developed almost at the same time: the model of Chesney and Taschini (2008) and the model of Carmona et al. (2009b). All these stochastic equilibrium models include the costs from a possible penalty payment in the target function of the optimization problem. Therefore, stochastic equilibrium models capture the main characteristics of an ordinary scheme (cf. Section 1.1).

Two minor aspects of the approach of Seifert et al. (2008) are different from the modelling approaches of the other equilibrium models:

1. Emissions are broken down into emissions before abatement activities and abated emissions (in the other models emissions are not split up).
2. Permit price dynamics are derived for a representative agent with the aim to simplify the notation (in the other models the permit price formula follows directly from the individual and joint optimization problem).

This subsection is split up into two parts:

- It is shown that the market equilibrium is equivalent to the joint cost optimum. A by-product of this proof is that the permit price equals marginal abatement costs (cf. Subsection "Cost optimality of emissions").
- The dynamics of the permit price are derived by solving the cost minimization problem of a so-called representative agent. The introduction of a representative agent significantly simplifies the notation and is justified by the equivalence of the market equilibrium and the joint cost optimum (cf. Subsection "Permit price dynamics").

Cost optimality of emissions trading

The model of Seifert et al. (2008) assumes that there are n companies. Interest rates, r, are deterministic.
Emissions of firm i are modelled by

- β_t^i (stochastic process): emission rate before abatement activities
  Seifert et al. (2008) assume that for constant β_0^i and σ_i^2, the emission rate β_t^i either follows
  - a White-Noise process, i.e. $\beta_t \sim N(\beta_0, \sigma^2)$ or
  - an arithmetic Brownian motion of the form $\beta_t = \beta_0 + \sigma W_t$.
- α_t^i (stochastic process): abatement rate

Emissions trading is taken into account by

- θ_t^i (stochastic process): instantaneous amount of permits bought from or sold to other companies. Positive/Negative values correspond to buying/selling.

The ability to comply at the end of the compliance period depends both on the expected number of emissions in the interval $[0, T]$, the abatement volume and on the number of purchased permits:

$$q_t^i = \mathbb{E}\left[\int_0^t \beta_s^i ds \mid \mathcal{F}_t\right] - \int_0^t \alpha_s^i ds - \int_0^t \theta_s^i ds.$$

This means that q_t^i is equal to total cumulative emissions minus the number of permits bought from other companies.

Before the start of the compliance period, the regulator defines the following two parameters

- N^i: number of permits allocated to firm i at the beginning
- P: penalty that has to be paid for each unit of emission that is not covered by an emission allowance at the end of the compliance period

In the model of Seifert et al. (2008), an agent faces three different types of costs:

1. Abatement costs
   Seifert et al. (2008) model marginal abatement costs as a deterministic function that is increasing in the abatement volume. More precisely, instantaneous abatement costs at time t are modelled by
   $$C^i(\alpha_t^i) = \frac{1}{2}c_i(\alpha_t^i)^2,$$
   where c_i is a positive constant.
   The main difference between the models of Carmona et al. (2009b) and Seifert et al. (2008) is the different treatment of abatement costs: the model of Carmona et al. (2009b) assumes stochastic abatement costs that do not explicitly depend on the abated volume whereas the model of Seifert et al. (2008) assumes that there is a deterministic relationship between abatement costs and the abated volume.

2. Costs from permit purchases
   Firm i has the option to buy permits instead of abating or to sell permits to other companies if it is able to reduce emissions at low costs. Instantaneous costs/profits from emissions trading at time t are equal to
   $$S_t\theta_t^i,$$
   where S_t is the permit price at time t.

3. Penalty payment
   At the end of the compliance period, firm i has to pay the penalty P per unit of emission that is not covered by a permit, i.e. the total penalty equals
   $$P\left(q_T^i - N^i\right)^+.$$

Definition 3.2.1 (Firm i's optimization problem)
Given the permit price S_t, firm i minimizes its expected costs by choosing an optimal abatement strategy and buying or selling an optimal number of permits, i.e.

$$\begin{aligned}
&\max_{\alpha_t^i,\theta_t^i} \mathbb{E}\left[-\int_0^T e^{-rt}C^i(\alpha_t^i)dt - \int_0^T e^{-rt}S_t\theta_t^i - e^{-rT}P(q_T^i - N^i)^+\right] \qquad (3.38)\\
&= \max_{\alpha_t^i,\theta_t^i} \mathbb{E}\left[-\int_0^T e^{-rt}\left(\frac{1}{2}c_i(\alpha_t^i)^2\right)dt - \int_0^T e^{-rt}S_t\theta_t^i - e^{-rT}P(q_T^i - N^i)^+\right].
\end{aligned}$$

Lemma 3.2.2 (SDE for firm i's cumulative "emissions" q_t^i)
Assume that the emission rate before abatement activities, β_t^i, follows (i) a White-Noise process or (ii) an arithmetic Brownian motion.
Then the SDE of firm i's cumulative emissions are given by

$$dq_t^i = -(\alpha_t^i + \theta_t^i)dt + H_t^i dW_t,$$

where H_t^i is given in case (i) by $H_t^i = \sigma_i$ and in case (ii) by $H_t^i = \sigma_i(T-t)$.

Proof :
See online appendix of Seifert et al. (2008). ♢

Lemma 3.2.3 (First order conditions of firm i's optimization problem)
Let $V^i(t, q_t^i)$ be the expected value of an optimal policy for firm i's cost minimization problem (cf. Definition 3.2.1) between time t and T given q_t^i. Denote its partial derivatives by V_t^i, V_q^i, V_{qq}^i. Then the first order conditions of this optimization problem are given by

$$\begin{aligned} \alpha_t^i &= -\frac{1}{c} e^{rt} V_q^i \\ S_t &= -e^{rt} V_q^i = c_i \alpha_t^i = -\frac{\partial C^i(\alpha_t^i)}{\partial \alpha_t^i} \end{aligned}$$

Proof (Idea):
The proof is an adaption of the proof in Chapter 12.4 of Sethi and Thompson (1981). By the principle of optimality

$$V^i(t, q_t^i) = \max_{\alpha_t^i, \theta_t^i} \mathbb{E}\left[-e^{-rt} C^i(\alpha_t^i)dt - e^{-rt} S_t \theta_t^i dt + V^i(t+dt, q_t^i + dq_t^i)\right]. \tag{3.39}$$

Applying Itô's lemma to $V^i(t, q_t^i)$, we have

$$\begin{aligned} V^i(t+dt, q_t^i + dq_t^i) - V^i(t, q_t^i) &= V_t^i dt + V_q^i dq_t^i + \frac{1}{2} V_{qq}^i dq_t^i dq_t^i \\ &= V_t^i dt - V_q^i(\alpha_t^i + \theta_t^i)dt + H_t^i V_q^i dW_t + \frac{1}{2}(H_t^i)^2 V_{qq}^i dt \\ &= \left(V_t^i - V_q^i(\alpha_t^i + \theta_t^i) + \frac{1}{2}(H_t^i)^2 V_{qq}^i\right) dt + H_t^i V_q^i dW_t. \end{aligned}$$

This implies

$$\mathbb{E}[V^i(t+dt, q_t^i + dq_t^i) - V^i(t, q_t^i) \mid \mathcal{F}_t] = \left(V_t^i - V_q^i(\alpha_t^i + \theta_t^i) + \frac{1}{2}(H_t^i)^2 V_{qq}^i\right) dt.$$

Subtracting $V^i(t, q_t^i)$ on both sides of Equation (3.39) yields

$$0 = \max_{\alpha_t^i, \theta_t^i} \left[-e^{-rt} C^i(\alpha_t^i) - e^{-rt} S_t \theta_t^i + V_t^i - V_q^i(\alpha_t^i + \theta_t^i) + \frac{1}{2}(H_t^i)^2 V_{qq}^i \right], \tag{3.40}$$

with boundary condition

$$V^i(T, q_T^i) = -e^{-rT} P(q_T^i - N^i)^+.$$

Maximizing the right hand side of Equation (3.40) by taking the partial derivatives with respect to α_t^i and θ_t^i and setting it to zero yields

$$\begin{aligned} \alpha_t^i &= -\frac{1}{c_i} e^{rt} V_q^i \\ S_t &= -e^{rt} V_q^i = c_i \alpha_t^i = \frac{\partial C^i(t, \alpha_t^i)}{\partial \alpha_t^i}. \end{aligned}$$

$\diamondsuit$

Definition 3.2.4 (Market equilibrium)
A market equilibrium with associated optimal strategies, consisting of $\bar{S}_t$, $(\bar{\alpha}_t^1, \ldots, \bar{\alpha}_t^n)$ and $(\bar{\theta}_t^1, \ldots, \bar{\theta}_t^n)$, solves the firms' individual cost optimization problems as given in Definition 3.2.1 and satisfies the market clearing condition

$$\sum_{i=1}^n \bar{\theta}_t^i = 0.$$

Definition 3.2.5 (Global optimization problem)
The central planner minimizes joint costs of the firms by choosing optimal abatement strategies:

$$\max_{\alpha_t^1, \ldots, \alpha_t^n} \mathbb{E} \left[-\int_0^T e^{-rt} \sum_{i=1}^n \left(\frac{1}{2} c_i (\alpha_t^i)^2 \right) dt - e^{-rT} P \sum_{i=1}^n (q_T^i - N^i)^+ \right]. \tag{3.41}$$

Remark:
Seifert et al. (2008) show that the market equilibrium is also a solution of the joint cost problem and vice versa (cf. online appendix of their paper). This equivalence enables us to focus on the cost optimization problem of a representative agent in the following.

Permit price dynamics

The price dynamics result from the cost minimization problem of a so-called representative agent. The notation in the framework of the representative agent is almost the same as the notation of the previous subsection. The only minor difference is the variable q_t. In the previous subsection, q_t was defined as total cumulative emissions minus the number of permits bought from other companies whereas in the framework of the representative agent q_t equals total cumulative emissions:

$$q_t = \mathbb{E}\left[\int_0^t \beta_s ds \mid \mathcal{F}_t\right] - \int_0^t \alpha_s ds.$$

The dynamics of the total cumulative emissions are given in Lemma 3.2.6. A characteristic partial differential equation (PDE) and an analytical expression for the permit price is to be found in Theorem 3.2.8.

Lemma 3.2.6 (SDE for emissions of the representative agent q_t)
Assume that the emission rate before abatement activities, β_t, follows
(i) the White-Noise process $\beta_t \sim N(\beta_0, \sigma^2)$ or
(ii) the arithmetic Brownian motion $\beta_t = \beta_0 + \sigma W_t$.
Then the SDE for the cumulative emissions of the representative agent are given by

$$dq_t = -\alpha_t dt + H_t dW_t,$$

where H_t is given in case (i) by $H_t = \sigma$ and in case (ii) by $H_t = \sigma(T-t)$.

Proof :
See online appendix of Seifert et al. (2008). $\diamondsuit$

Definition 3.2.7 (Optimization problem of the representative agent)
Given the permit price S, the representative agent minimizes its expected costs by choosing an optimal abatement strategy:

$$\begin{aligned}&\max_{\alpha_t} \mathbb{E}\left[-\int_0^T e^{-rt} C(\alpha_t) dt - e^{-rT} P(q_T - N)^+\right] \qquad (3.42)\\ &= \max_{\alpha_t} \mathbb{E}\left[-\int_0^T e^{-rt}\left(\frac{1}{2}c(\alpha_t)^2\right) dt - e^{-rT} P(q_T - N)^+\right].\end{aligned}$$

Theorem 3.2.8 (Permit price dynamics)
Let $V(t, q_t)$ be the expected value of an optimal policy for the optimization problem in Definition 3.2.7 between time t and T given q_t.
Denote its partial derivatives by V_t, V_q, V_{qq}.
(a) Assume that the emission rate before abatement activities is given by the arithmetic Brownian motion in Lemma 3.2.6.
Then the characteristic PDE of the permit price is given by

$$V_t + \frac{1}{2}\sigma^2(T-t)^2 V_{qq} + \frac{1}{2c} e^{rt} \left(V_q\right)^2 = 0,$$

with boundary condition

$$V(T, q_T) = e^{rt} P \left(q_T - N\right)^+,$$

and the permit price is given by

$$S_t = -e^{rt} V_q.$$

(b) Assuming that the emission rate before abatement activities is given by the White-Noise process in Lemma 3.2.6. Then the characteristic PDE of the permit price is given by

$$V_t + \frac{1}{2}\sigma^2 V_{qq} + \frac{1}{2c} e^{rt} \left(V_q\right)^2 = 0,$$

and there is an analytical formula for the permit price:

$$S(t, q_t) = P \cdot \frac{1}{1 - \frac{\exp\left\{\frac{-P[P(T-t)+2c(N-q_t)]}{2c^2\sigma^2}\right\}\left(-2 + erfc\left(\frac{N-q_t}{\sigma\sqrt{2(T-t)}}\right)\right)}{erfc\left(\frac{P(T-t)+c(N-q_t)}{\sigma\sqrt{2(T-t)}}\right)}},$$

where $erfc(x) = 1 - erf(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$ is the complementary error function.

Proof (Idea):
Similar to the proof of Lemma 3.2.3, Itô's lemma and Lemma 3.2.6 imply

$$V(t+dt, q_t + dq_t) - V(t, q_t) = \left(V_t - \alpha_t V_q + \frac{1}{2}(H_t)^2 V_{qq}\right) dt + H_t V_q dW_t,$$

$$\mathbb{E}[V(t+dt, q_t + dq_t) - V(t, q_t) \mid \mathcal{F}_t] = V_t dt - \alpha_t V_q dt + \frac{1}{2}(H_t)^2 V_{qq} dt.$$

By the principle of optimality

$$V(t, q_t) = \max_{\alpha_t} \mathbb{E}\left[-e^{-rt} C(\alpha_t) dt + V(t+dt, q_t + dq_t)\right]. \tag{3.43}$$

Subtracting $V(t, q_t)$ on both sides of Equation (3.43) yields

$$0 = \max_{\alpha_t} \left\{ -e^{-rt} C(\alpha_t) + V_t - \alpha_t V_q + \frac{1}{2}(H_t)^2 V_{qq} \right\}. \tag{3.44}$$

Maximizing the expression within the curly brackets by deriving it with respect to α_t and setting it to zero yields

$$\alpha_t = -\frac{1}{c} e^{rt} V_q, \quad \text{or} \quad c\alpha_t = -e^{rt} V_q. \tag{3.45}$$

The characteristic PDE is obtained by setting the expression within the curly brackets in Equation (3.44) to zero and by inserting the formula for α_t into this equation:

$$\begin{aligned}
& - e^{-rt} C(\alpha_t) dt + V_t - \alpha_t V_q + \frac{1}{2}(H_t)^2 V_{qq} = 0 \\
\Leftrightarrow \; & - e^{-rt} \frac{1}{2} c(\alpha_t)^2 + V_t - \alpha_t V_q + \frac{1}{2}(H_t)^2 V_{qq} = 0 \\
\Leftrightarrow \; & - \frac{1}{2c} e^{rt} (V_q)^2 + V_t + \frac{1}{c} e^{rt} (V_q)^2 + \frac{1}{2}(H_t)^2 V_{qq} = 0 \\
\Leftrightarrow \; & V_t + \frac{1}{2c} e^{-rt} (V_q)^2 + \frac{1}{2}(H_t)^2 V_{qq} = 0.
\end{aligned} \tag{3.46}$$

The spot price equals marginal abatement costs as shown in Lemma 3.2.3, i.e.

$$S_t = \frac{\partial C(\alpha_t)}{\partial \alpha_t} = c\alpha_t \overset{(3.45)}{=} -e^{-rt} V_q.$$

(a) The proof is completed by using Lemma 3.2.6 (ii).

(b) The characteristic PDE follows directly from Equation (3.46) by using part (i) of Lemma 3.2.6. A solution of this PDE is derived by Seifert et al. (2008). ♢

Remark:

If the emission rate follows an arithmetic Brownian motion, it is not possible to derive a closed-form solution due to the more complex PDE. Seifert et al. (2008) present graphical illustrations of the permit price in this case.

Results

The graphical illustrations of Seifert et al. (2008) show that the permit price in the model of Seifert et al. (2008) is similar to the value of P binary call options with underlying q_T, strike price N and expiry time T. Figure 3.1 illustrates the payoff of a binary call option at expiry time and the value of this option before expiry time. A definition of a binary option is to be found in Definition 3.2.9. As the value of a binary option can be interpreted as the probability that the price of the underlying at time T is greater than the strike price, the permit price in the model of Seifert et al. (2008) can be interpreted as the penalty P multiplied by the probability that cumulative emissions in the compliance period are greater than the number of allowances, N.
Moreover, Seifert et al. (2008) show that the permit price equals marginal abatement costs. Combining this equality and the discussion about digital options yields an extremely interesting result: permit prices can be interpreted both as the penalty multiplied by the probability of permit shortage and as marginal abatement costs. In other words, the concept of probability of permit shortage is an extension of the concept of marginal abatement costs that explicitly takes the dependency of the permit price on the penalty fee, the number of allocated permits and on the expectation of future emissions into account. Therefore, the paper of Seifert et al. (2008) links approaches from two different fields:

- Environmental Economics (marginal abatement costs)
- Financial Mathematics (probability of permit shortage).

Definition 3.2.9 (Binary option)
A binary call option (also known as digital call option) with expiry time T and strike price K pays 1 Euro at time T if the price of the underlying S_T is greater than the strike price K. The payoff at time T in mathematical notation is given by

$$\mathbf{1}_{\{S_T > K\}}.$$

Remark:
The value of the binary call option before expiry time is equal to the probability that the price of the underlying at time T is greater than the strike price.

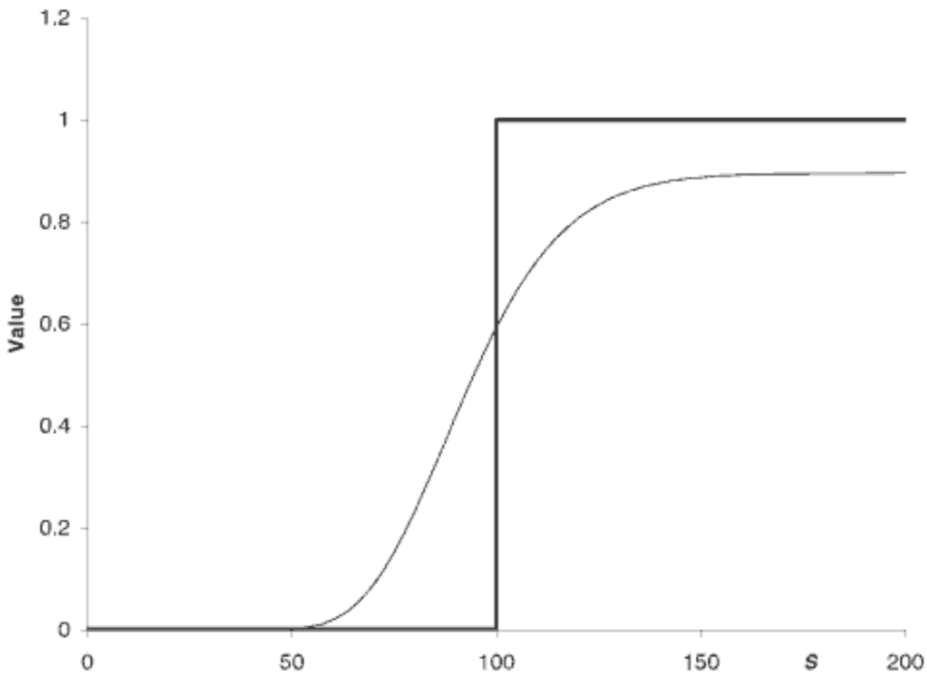

Figure 3.1: Illustration of a binary call option with strike price $K = 100$: bold line is the payoff at expiry time and the fine line is the contract value some time before expiry. Source: Wilmott (2007).

3.2b Model of Carmona et al. (2009b)

Introduction

The stochastic equilibrium model of Carmona et al. (2009b) is of special interest because:

- The model captures the main characteristics of an ordinary scheme and
- Within the general framework of Carmona et al. (2009b), it is possible to derive analytically tractable permit price formulae as shown in Section 3.2c and 3.2d and
- The model of Carmona et al. (2009b) is the basis for the reduced-form models that can be used for parameter estimation in practice.

Definitions

The model of Carmona et al. (2009b) assumes that there are n companies. Note that the optimization problems of Carmona et al. (2009b) are all formulated in a way such

that profits and costs are expressed in time-T currency. This has the advantage that no discount factors appear in the lengthy formulae of the optimization problems.
It is assumed that firms maximize their profits by producing an optimal amount of goods and choosing an optimal permit trading strategy.
The target function of the optimization problem consists of three parts

1. Production of goods
   Firm i's profit from producing goods over the time period $[0,T]$ is equal to

   $$R^i_{\text{good}}(y^i) - C^i_{\text{good}}(y^i) = \sum_{t=0}^{T-1}\left(\sum_{j,k} G^k_t y^{i,j,k}_t\right) - \sum_{t=0}^{T-1}\left(\sum_{j,k} \kappa^{i,j,k}_t y^{i,j,k}_t\right),$$

   where

   - $y^{i,j,k}_t$ (stochastic process) models firm i's output quantity of good k at time t when using technology j
   - G^k_t (stochastic process) models the price of one unit of good k at time t. Prices are expressed in time-T currency.
   - $\kappa^{i,j,k}_t$ (stochastic process) models firm i's marginal costs of producing one unit of good k when using technology j. Costs are expressed in time-T currency. It is assumed that marginal costs are given exogenously.

2. Trading of emission allowances
   Gain or loss from trading of emission allowances is equal to

   $$T^i(\Theta^i) = \sum_{t=0}^{T-1} \Theta^i_t \left(A_{t+1} - A_t\right) - \Theta^i_T A_T,$$

   where

   - $\Theta^{i,j,k}_t$ (stochastic process) models the total number of permits that firm i bought from or sold to other companies until time t. Positive/Negative values indicate that firm i is a net buyer/seller in the period $[0,t]$.
   - A_t (stochastic process) models the time-t price of the futures on the emission allowance maturing at time T

3. Penalty payment at the end of the compliance period
   The total penalty of firm i is given by

   $$P \cdot \left(q^i(y^i) + \Delta^i - N^i - \Theta^i_T\right)^+$$

   and it depends both on

(a) firm i's cumulative emissions in the time period $[0, T]$

$$q^i(y^i) + \Delta^i = \sum_{t=0}^{T-1} Q_t^i + \Delta^i = \sum_{t=0}^{T-1} \left(\sum_{j,k} e^{i,j,k} y_t^{i,j,k} \right) + \Delta^i,$$

where

- $e^{i,j,k}$ (constant) is the emission factor measuring the emissions that are caused when firm i is producing one unit of good k with technology j
- Δ^i (random variable) models emissions that firm i cannot control

(b) parameters of the scheme as defined by the regulator

- P (constant) is the penalty fee per unit of emission that is not covered by an emission allowance at the compliance time T
- $N^i = \sum_{t=0}^{T-1} N_t^i$ (random variable) models the number of allowances that the regulator allocates to firm i in the compliance period $[0, T]$

Remark:
(a) Production of goods has to satisfy the following two constraints

1. Production cannot exceed capacity:

$$0 \leq y_t^{i,j,k} \leq K^{i,j,k}.$$

2. Demand is always smaller than the total production capacity:

$$0 \leq D_t^k \leq \sum_i \sum_j K^{i,j,k},$$

where

- D_t^k (stochastic process) models the demand for good k at time t
- $K^{i,j,k}$ (constant) models firm i's capacity constraint to produce good k with technology j

(b) The optimization problem of Carmona et al. (2009b) models trading with the help of Θ_t^i, the total number of traded permits that firm i holds at time t, instead of the number of permits that firm i buys or sells at time (denoted by θ_t^i). The reason is that it is easier to compute the penalty in terms of Θ_t^i than in terms of θ_t^i. However, it is possible to transform Θ_t^i into θ_t^i:

$$\theta_0^i = \Theta_0^i,$$
$$\theta_t^i = \Theta_t^i - \Theta_{t-1}^i.$$

Gain or loss from trading activities can be expressed as follows

$$\begin{aligned}
T^i(\Theta^i) = & -\sum_{t=0}^{T} \theta_t^i A_t \\
= & -\Theta_0^i A_0 - \sum_{t=1}^{T} \left(\Theta_t^i - \Theta_{t-1}^i\right) A_t \\
= & -\Theta_0^i A_0 + \Theta_0^i A_1 - \Theta_1^i A_1 + \Theta_1^i A_2 - \Theta_2^i A_2 + \ldots + \Theta_{T-2}^i A_{T-1} - \Theta_{T-1}^i A_{T-1} \\
& + \Theta_{T-2}^i A_T - \Theta_T^i A_T \\
= & \Theta_0^i (A_1 - A_0) + \Theta_1^i (A_2 - A_1) + \ldots + \Theta_{T-1}^i (A_T - A_{T-1}) - \Theta_T^i A_T \\
= & \sum_{t=0}^{T-1} \Theta_t^i \left(A_{t+1} - A_t\right) - \Theta_T^i A_T.
\end{aligned}$$

(c) The uncontrollable emissions have to satisfy the following condition: conditional on the information available at time $T-1$, the sum of all uncontrollable emissions, $\sum_i \Delta^i$, must have a continuous distribution. This technical assumption is introduced in order to avoid pathological situations concerning the equilibrium prices.

Optimization problems

Definition 3.2.10 (Firm i's optimization problem)
Given the forward permit price A and the prices of the produced goods G, firm i maximizes its expected terminal wealth by buying or selling an optimal number of permits and producing an optimal quantity of goods, i.e.

$$\sup_{\Theta^i, y^i} \mathbb{E}\left[L^i\left(\Theta^i, y^i \mid A, G\right)\right] \tag{3.47}$$

where the terminal wealth is given by

$$\begin{aligned}
& L^i\left(\Theta^i, y^i \mid A, G\right) \\
& \quad = \left[R_{good}^i(y^i \mid G) - C_{good}^i(y^i) + T^i(\Theta^i \mid A) - P \cdot \left(q^i(y^i) + \Delta^i - N^i - \Theta_T^i\right)^+\right].
\end{aligned} \tag{3.48}$$

Definition 3.2.11 (Global optimization problem)
A fictitious central planner minimizes expected total costs by producing an optimal quantity of goods, i.e. it faces the following optimization problem

$$\inf_y \mathbb{E}\left[C_{good}(y) + P\left(q(y) + \Delta - N\right)^+\right], \tag{3.49}$$

where

$$\begin{aligned} C_{good}(y) &= \sum_i C^i_{good}(y^i), \\ q(y) &= \sum_i q^i(y^i), \\ \Delta &= \sum_i \Delta^i, \\ N &= \sum_i N^i. \end{aligned}$$

Relationship between market equilibrium and global optimum

Definition 3.2.12 (Market equilibrium)
$(\bar{A}, \bar{G})$ is a market equilibrium in emission permits with associated strategies $\bar{\Theta}$ and $\bar{y}$ if for given

- $\bar{A}$ *(one-dimensional stochastic process for forward price on permits)*
- $\bar{G}$ *(multi-dimensional stochastic process for the prices of the products)*

the associated optimal strategies

- $\bar{\Theta}$ *(multi-dimensional stochastic process of optimal trading strategies)*
- $\bar{y}$ *(multi-dimensional stochastic process of optimal production strategies)*

lead to a situation where all the firms ("maximize" their profits) are satisfied by their strategy in the sense that for all i

$$\mathbb{E}\left[L^i\left(\bar{\Theta}^i, \bar{y}^i \mid \bar{A}, \bar{G}\right)\right] \geq \mathbb{E}\left[L^i\left(\Theta^i, y^i \mid \bar{A}, \bar{G}\right)\right] \text{ for all } \left(\Theta^i, y^i\right)$$

and the following two conditions hold

- *Market clearing condition on permits*

$$\sum_i \bar{\Theta}_t^i = 0$$

- *Supply meets demand for each good*

$$\sum_{i,j} \bar{y}_t^{i,j,k} = D_t^k$$

Lemma 3.2.13 (Market equilibrium and joint optimization problem)
If $(\bar{A}, \bar{G})$ *is a market equilibrium with associated strategies* $(\bar{\Theta}, \bar{y})$ *then*
(a) the permit price and the prices of the goods are almost surely given by

$$\begin{aligned}\bar{A}_t &= P \cdot \mathbb{P}\left(q(\bar{y}) + \Delta \geq N \mid \mathcal{F}_t\right), \\ \bar{G}_t &= \max_{i,j}\left\{\left(\kappa_t^{i,j,k} + e^{i,j,k}\bar{A}_t\right)\boldsymbol{1}_{\{\bar{y}_t^{i,j,k}>0\}}\right\}.\end{aligned}$$

(b) $\bar{y}$ *is a solution of the global optimization problem.*

Remark:
(a) In the business as usual scenario (i.e. $P = 0$), the equilibrium prices of the goods are given by

$$\bar{G}_t = \max_{i,j}\left\{\kappa_t^{i,j,k}\mathbf{1}_{\{\bar{y}_t^{i,j,k}>0\}}\right\}.$$

This means that equilibrium prices correspond to a merit-order type equilibrium in which

- all the production means of the economy are ranked by increasing production costs, $\kappa_t^{i,j,k}$
- demand is met by producing from the cheapest production means
- the equilibrium price of good k is equal to the marginal cost of production of the most expensive production means used to meet demand D_t^k

(b) In the presence of a penalty $P > 0$, the equilibrium prices of the goods are given by

$$\bar{G}_t = \max_{i,j}\left\{\left(\kappa_t^{i,j,k} + e^{i,j,k}\bar{A}_t\right)\mathbf{1}_{\{\bar{y}_t^{i,j,k}>0\}}\right\}.$$

This means that equilibrium prices correspond to a merit-order type equilibrium with adjusted costs $\kappa_t^{i,j,k} + e^{i,j,k}\bar{A}_t$.

Lemma 3.2.14 (Joint optimization problem and market equilibrium)
(a) There exists a solution $\tilde{y}$ of the global optimization problem.
(b) If $\tilde{y}$ is a a solution of the global optimization problem then

(i) the permit price and the prices of the goods are almost surely given by

$$\tilde{A}_t = P \cdot \mathbb{P}\left(q(\tilde{y}) + \Delta \geq N \mid \mathcal{F}_t\right),$$
$$\tilde{G}_t = \max_{i,j} \left\{ \left(\kappa_t^{i,j,k} + e^{i,j,k} \tilde{A}_t \right) \mathbf{1}_{\left\{ \tilde{y}_t^{i,j,k} > 0 \right\}} \right\}.$$

and the allowance price process is almost surely unique.

(ii) $(\tilde{A}, \tilde{G})$ is a market equilibrium.

Results

In the model of Carmona et al. (2009b)

- The market equilibrium is equivalent to the joint cost optimum.
- The futures permit price is equal to the penalty multiplied by the probability of permit shortage at the end of the compliance period. The event of permit shortage is defined as the situation that cumulative emissions (after abatement activities) of all regulated companies in the compliance period exceed the total number of permits.
- Prices of the produced goods correspond to a merit-order type equilibrium with adjusted costs. This means that products are becoming more expensive in the presence of an emissions trading scheme. The price increase is equal to the value of the permits that are needed for the production of the good.

3.2c Model of Chesney and Taschini (2008)

Introduction

Even though the model of Chesney and Taschini (2008) was developed independently from the model of Carmona et al. (2009b), it can be categorized as a model in the general framework of Carmona et al. (2009b). The model of Chesney and Taschini (2008) specifies the process for the cumulative emissions in the framework of Carmona et al. (2009b).

Notation

The following notation is used in the model of Chesney and Taschini (2008), in the extended model of Chesney and Taschini (2008) and in the model of Grüll and Kiesel (2009) (cf. Section 3.2c).

- $q_{[0,T]}$ is the random variable that denotes the aggregated cumulative emissions of all regulated companies at time T.
- P is the penalty fee that has to be paid for each emission unit not covered by an emission allowance at the compliance time T.
- N is the total amount of permits allocated by the policy regulator to relevant companies, i.e. the cap.

Allowing for stochastic production costs, revenues from selling produced goods and emission quantities, Carmona et al. (2009b) prove in a general setting that the time-t futures price (with maturity T) for emission permits is given by the penalty multiplied by the probability of a permit shortage at the end of the compliance period, i.e.

$$F(t,T) = P \cdot \mathbb{P}\left(q_{[0,T]} > N|\mathcal{F}_t\right), \tag{3.50}$$

Assuming that interest rates r are deterministic and that there is no convenience yield as shown by Uhrig-Homburg and Wagner (2007) the theoretical permit price is given by

$$\begin{aligned} S_t &= Pe^{-r(T-t)} \cdot \mathbb{P}\left(q_{[0,T]} > N|\mathcal{F}_t\right) \\ &= \begin{cases} Pe^{-r(T-t)} & \text{if } q_{[0,t]} \geq N \\ Pe^{-r(T-t)} \cdot \mathbb{P}\left(q_{[t,T]} > N - q_{[0,t]}|\mathcal{F}_t\right) & \text{if } q_{[0,t]} < N \end{cases} \end{aligned} \tag{3.51}$$

Original model of Chesney and Taschini (2008)

The model of Chesney and Taschini (2008) specifies the process for the cumulative emissions in the framework of Carmona et al. (2009b) by assuming that the firms' emission rate, Q_t, follows a geometric Brownian motion, i.e.

$$Q_t = Q_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right\}. \tag{3.52}$$

Therefore, the cumulative emissions in $[0,t]$ are given by

$$q_{[0,t]} = \int_0^t Q_s ds. \tag{3.53}$$

This means that the cumulative emissions are described by the integral over a geometric Brownian motion for which no closed-form density is available. The model of Chesney and Taschini (2008) approximates the cumulative emissions in the time interval $[t_1, t_2] \subseteq [0,T]$ by the following linear approximation

$$q_{[t_1,t_2]} \approx \tilde{q}^{Lin}_{[t_1,t_2]} = Q_{t_2}(t_2 - t_1). \tag{3.54}$$

Lemma 3.2.15 (Cumulative emissions in the model of Chesney and Taschini)
Let μ and σ be the parameters of the geometric Brownian motion modelling the emission rate. Let $t \in [0,T]$, $\tau = T - t$ and $Z \sim N(0,1)$.
Then the cumulative emissions in the interval $[t,T]$ are given by

$$\tilde{q}^{Lin}_{[t,T]} = Q_t \exp\left\{\ln(\tau) + \left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}Z\right\}. \tag{3.55}$$

Proof :

$$\begin{aligned}
\tilde{q}^{Lin}_{[t,T]} &\overset{(3.54)}{=} (T-t)\cdot Q_T = \tau \cdot Q_T \\
&\overset{(3.60)}{=} \tau \cdot Q_t \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma W_\tau\right\} \\
&= Q_t \exp\left\{\ln(\tau) + \left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}Z\right\}.
\end{aligned}$$

$\diamondsuit$

Lemma 3.2.16 (Permit price - Linear approximation)
The permit price at time $t < T$ *is given by*

$$S_t^{Lin} = \begin{cases} Pe^{-r\tau} & \textit{if } q_{[0,t]} \geq N \\ Pe^{-r\tau} \cdot \Phi\left(\frac{-\ln\left(\frac{1}{\tau}\left[\frac{N-q_{[0,t]}}{Q_t}\right]\right)+\left(\mu-\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right) & \textit{if } q_{[0,t]} < N \end{cases} \tag{3.56}$$

where $\tau = T - t$ *is the time to compliance.*

Remark:
The permit price at time T is given by

$$S_T^{Lin} = P \cdot \mathbf{1}_{\left\{q_{[0,T]} \geq N\right\}}.$$

Proof :

$$S_t^{Lin} \overset{(3.51),(3.54)}{=} \begin{cases} Pe^{-r\tau} & \text{if } q_{[0,t]} \geq N \\ Pe^{-r\tau} \cdot \mathbb{P}\left(\tau \cdot Q_T > N - q_{[0,t]} | \mathcal{F}_t\right) & \text{if } q_{[0,t]} < N \end{cases}$$

Let $Z \sim N(0,1)$. Then,

$$\begin{aligned} &\mathbb{P}\left(\tau \cdot Q_T > N - q_{[0,t]} | \mathcal{F}_t\right) \\ &\overset{(3.60)}{=} \mathbb{P}\left(\tau \cdot Q_t \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}Z\right\} > N - q_{[0,t]} | \mathcal{F}_t\right) \\ &\overset{Q_t > 0}{=} \mathbb{P}\left(\exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}Z\right\} > \frac{1}{\tau}\left[\frac{N - q_{[0,t]}}{Q_t}\right] | \mathcal{F}_t\right) \\ &\overset{N > q_{[0,t]}}{=} 1 - \Phi\left(\frac{\ln\left(\frac{1}{\tau}\left[\frac{N-q_{[0,t]}}{Q_t}\right]\right) - \left(\mu - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right) \\ &= \Phi\left(\frac{-\ln\left(\frac{1}{\tau}\left[\frac{N-q_{[0,t]}}{Q_t}\right]\right) + \left(\mu - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right) \end{aligned}$$

completes the proof. ♢

Extended model of Chesney and Taschini (2008)

The extended model of Chesney and Taschini (2008) was introduced by Grüll and Taschini (2009) to analyze the relationship between stochastic equilibrium models and the reduced-form model of Carmona et al. (2009a).
In the extended model of Chesney and Taschini (2008) it is assumed that the firm's emission rate, Q_t, follows a geometric Brownian motion with time-dependent drift and volatility, i.e.

$$Q_t = Q_0 \exp\left\{ \int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s) \right) ds + \int_0^t \sigma(s) dW_s \right\}. \tag{3.57}$$

This means that the cumulative emissions, $q_{[0,t]} = \int_0^t Q_s ds$, are described by an integral over a geometric Brownian motion with time-dependent drift and volatility for which no closed-form density is available. Similar to the original model of Chesney and Taschini (2008), cumulative emissions are approximated linearly (cf. Equation (3.54)). The resulting process is given in Lemma 3.2.17.

Lemma 3.2.17 (Cumulative emissions in the extended model of CT)
Assume that the emission rate, Q_t, is modelled by a geometric Brownian motion with time-dependent deterministic drift, $\mu(s)$, and volatility, $\sigma(s)$.
Let $t \in [0,T]$ and $Z \sim N(0,1)$.
Then the cumulative emissions in the interval $[t,T]$ are given by

$$\tilde{q}_{[t,T]}^{Lin-Ext} = Q_t \exp\left\{ \ln(T-t) + \int_t^T \left(\mu(s) - \frac{1}{2}\sigma^2(s) \right) ds + \sqrt{\int_t^T \sigma^2(s) ds} \cdot Z \right\}. \tag{3.58}$$

Proof :

$$\begin{aligned}
\tilde{q}_{[t,T]}^{Lin-Ext} &\overset{(3.54)}{=} (T-t) \cdot Q_T \\
&\overset{(3.57)}{=} (T-t) \cdot Q_t \exp\left\{ \int_t^T \left(\mu(s) - \frac{1}{2}\sigma^2(s) \right) ds + \int_t^T \sigma(s) dW_s \right\} \\
&= Q_t \exp\left\{ \ln(T-t) + \int_t^T \left(\mu(s) - \frac{1}{2}\sigma^2(s) \right) ds + \int_t^T \sigma(s) dW_s \right\} \\
&= Q_t \exp\left\{ \ln(T-t) + \int_t^T \left(\mu(s) - \frac{1}{2}\sigma^2(s) \right) ds + \sqrt{\int_t^T \sigma^2(s) ds} \cdot Z \right\}.
\end{aligned}$$

$\diamondsuit$

Lemma 3.2.18 (Permit price in the extended model of Chesney and Taschini)

The permit price at time $t < T$ *is given by*

$$S_t^{Lin-Ext} = \begin{cases} Pe^{-r\tau} & \textit{if } q_{[0,t]} \geq N \\ Pe^{-r\tau} \cdot \Phi\left(\frac{-\ln\left(\frac{N-q_{[0,t]}}{\tau \cdot Q_t}\right) + \int_t^T \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right) ds}{\sqrt{\int_t^T \sigma^2(s) ds}}\right) & \textit{if } q_{[0,t]} < N \end{cases} \tag{3.59}$$

where $\tau = T - t$ *is the time to compliance.*

Proof :

$$S_t^{Lin-Ext} \overset{(3.51),(3.54)}{=} \begin{cases} Pe^{-r\tau} & \text{if } q_{[0,t]} \geq N \\ Pe^{-r\tau} \cdot \mathbb{P}\left(\tau \cdot Q_T > N - q_{[0,t]} | \mathcal{F}_t\right) & \text{if } q_{[0,t]} < N \end{cases}$$

Let $Z \sim N(0,1)$. Using Equation (3.57) yields

$$\begin{aligned} &\mathbb{P}\left(Q_t \exp\left\{\ln(\tau) + \int_t^T \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right) ds + \int_t^T \sigma(s) dW_s\right\} > N - q_{[0,t]} | \mathcal{F}_t\right) \\ &= \mathbb{P}\left(\exp\left\{\ln(\tau) + \int_t^T \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right) ds + \int_t^T \sigma(s) dW_s\right\} > \frac{N - q_{[0,t]}}{Q_t} | \mathcal{F}_t\right) \\ &= \mathbb{P}\left(\int_t^T \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right) ds + \int_t^T \sigma(s) dW_s > \ln\left(\frac{N - q_{[0,t]}}{\tau \cdot Q_t}\right) | \mathcal{F}_t\right) \\ &= \mathbb{P}\left(\sqrt{\int_t^T \sigma^2(s) ds} \cdot Z > \ln\left(\frac{N - q_{[0,t]}}{\tau \cdot Q_t}\right) - \int_t^T \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right) ds | \mathcal{F}_t\right) \\ &= \Phi\left(\frac{-\ln\left(\frac{N-q_{[0,t]}}{\tau \cdot Q_t}\right) + \int_t^T \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right) ds}{\sqrt{\int_t^T \sigma^2(s) ds}}\right) \end{aligned}$$

which completes the proof. ♢

3.2d Model of Grüll and Kiesel (2009)

Introduction

The model of Grüll and Kiesel (2009) is an extension of the model of Chesney and Taschini (2008). An overview of the notation can be found at the beginning of Section 3.2c. Both models specify the process for the cumulative emissions in the framework of Carmona et al. (2009b) by assuming that the firms' emission rate, Q_t , follows a geometric Brownian motion, i.e.

$$Q_t = Q_0 \exp\left\{ \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t \right\}. \tag{3.60}$$

Therefore, the cumulative emissions in $[0,t]$ are given by

$$q_{[0,t]} = \int_0^t Q_s ds. \tag{3.61}$$

This means that the cumulative emissions are described by the integral over a geometric Brownian motion for which no closed-form density is available. The models of Chesney and Taschini (2008) and Grüll and Kiesel (2009) differ in the way the cumulative emissions are approximated. The linear approximation approach of Chesney and Taschini (2008) (cf. Equation (3.54) in Section 3.2c) has the shortcoming that the moments of the approximated cumulative emissions do not match the true ones. Grüll and Kiesel (2009) overcome this over-simplification by applying a moment matching approach.
The method of moment matching is chosen because it was successfully applied to a similar problem in a completely different area of financial mathematics. Milevsky and Posner (1998) price Asian options by using a moment matching approach. An Asian option is a path-dependent option involving the average price of the underlying in the time period $[t_1, t_2]$. If the price of the underlying S_t is modelled by a geometric Brownian motion (common assumption in the framework of Black Scholes), the average price of the underlying,

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} S_t dt,$$

involves the integral over a geometric Brownian motion for which no closed-form density is available. The basic idea is to approximate this integral and to compute the price of the Asian option using the approximating random variable.

In the moment-matching approach the random variable of interest is approximated by a well-known and analytically tractable random variable. The parameters of the approximating random variable are chosen such that its first few moments match the moments of the random variable for which no closed-form density is available.

In the following we discuss two different moment-matching approaches for the cumulative emissions:

(a) Log-normal (moment matching)

$$q_{[t_1,t_2]} \approx \tilde{q}^{Log}_{[t_1,t_2]} = logN\left(\mu_L(t_1,t_2), \sigma^2_L(t_1,t_2)\right) \tag{3.62}$$

(b) Reciprocal gamma (moment matching)

$$q_{[t_1,t_2]} \approx \tilde{q}^{IG}_{[t_1,t_2]} = IG(\alpha_{IG}, \beta_{IG}) \tag{3.63}$$

where the parameters $\mu_L(t_1,t_2)$, $\sigma_L(t_1,t_2)$ and α_{IG} and β_{IG} are chosen such that the first two moments of $\tilde{q}^{Log}_{[t_1,t_2]}$ and $\tilde{q}^{IG}_{[t_1,t_2]}$, respectively, match those of $q_{[t_1,t_2]}$.

The derivation of the permit prices for the different approximation approaches is split up into three parts:

1. Compute the first two moments of $q_{[t_1,t_2]} = \int_{t_1}^{t_2} Q_s ds$ (cf. Lemma 3.2.19)
2. Derive the parameters of the random variables that are used to approximate the cumulative emissions (cf. Lemma 3.2.21)
3. Derive the permit price formulae for the two different moment matching approaches (cf. Lemma 3.2.22 and 3.2.23)

The derivation makes use of the properties of log-normal, Gamma and Reciprocal Gamma distributed random variables (cf. Lemma 3.2.24 - 3.2.26).

Numerical illustrations of the permit pricing formulae and interesting properties of the theoretical permit prices are presented in Section 4.1.

Moments of the cumulative emissions

Milevsky and Posner (1998) prove the following lemma for the integral over a geometric Brownian motion.

Lemma 3.2.19 (Moments of $q_{[t_1,t_2]}$)
Let $Q_s = Q_0 \exp\left\{ \left(\mu - \frac{\sigma^2}{2}\right) s + \sigma W_s \right\}$ *be a geometric Brownian motion.*
Then the first two moments of $q_{[t_1,t_2]} = \int_{t_1}^{t_2} Q_s ds$ *are given by*

$$m_1(t_1, t_2) := \mathbb{E}\left[q_{[t_1,t_2]}\right] = Q_{t_1} \alpha_{t_2-t_1}, \tag{3.64}$$

$$m_2(t_1, t_2) := \mathbb{E}\left[\left(q_{[t_1,t_2]}\right)^2\right] = 2Q_{t_1}^2 \beta_{t_2-t_1}, \tag{3.65}$$

where

$$\alpha_{t_2-t_1} = \begin{cases} \frac{1}{\mu}\left(e^{\mu(t_2-t_1)} - 1\right) & \text{if } \mu \neq 0 \\ t_2 - t_1 & \text{if } \mu = 0 \end{cases} \tag{3.66}$$

$$\beta_{t_2-t_1} = \begin{cases} \frac{\mu e^{(2\mu+\sigma^2)(t_2-t_1)} + \mu + \sigma^2 - \left(2\mu+\sigma^2\right) e^{\mu(t_2-t_1)}}{\mu(\mu+\sigma^2)(2\mu+\sigma^2)} & \text{if } \mu \neq 0 \\ \frac{1}{\sigma^4}\left(e^{\sigma^2(t_2-t_1)} - 1 - \sigma^2\left(t_2 - t_1\right)\right) & \text{if } \mu = 0 \end{cases} \tag{3.67}$$

Corollary 3.2.20 (Approximated moments of $q_{[t_1,t_2]}$)
For $\mu \neq 0$ *the Taylor expansions up to the third order are given by*

$$\begin{aligned} \alpha_{t_2-t_1} &\approx (t_2 - t_1) + \frac{\mu}{2}(t_2 - t_1)^2 + \frac{\mu^2}{6}(t_2 - t_1)^3, \\ \beta_{t_2-t_1} &\approx \frac{1}{2}(t_2 - t_1)^2 + \frac{3\mu^2 + \sigma^4 + 4\mu\sigma^2}{6(\mu + \sigma^2)} \cdot (t_2 - t_1)^3, \\ \mathbb{E}\left[q_{[t_1,t_2]}\right] &= Q_{t_1}\alpha_{t_2-t_1} \approx Q_{t_1}\left[(t_2 - t_1) + \frac{\mu}{2}(t_2 - t_1)^2 + \frac{\mu^2}{6}(t_2 - t_1)^3\right], \\ Var(q_{[t_1,t_2]}) &= Q_{t_1}^2\left(2\beta_{t_2-t_1} - \alpha_{t_2-t_1}^2\right) \approx Q_{t_1}^2 \cdot \frac{\sigma^2}{3}(t_2 - t_1)^3. \end{aligned}$$

Remark:
Corollary 3.2.20 shows that $\alpha_{t_2-t_1}$ is approximately linear and that $\beta_{t_2-t_1}$ is approximately a quadratic function for small μ and σ. Therefore, expected cumulative emissions are approximately linear.

Parameters of approximating random variables

Lemma 3.2.21 (Cumulative emissions in the model of Grüll and Kiesel (2009))
Let μ and σ be the parameters of the geometric Brownian motion modelling the emission rate. Let $t \in [0,T]$, $\tau = T - t$ and $Z \sim N(0,1)$.
Then the cumulative emissions in the interval $[t,T]$ are given by

(a) Log-normal (moment matching)

$$\tilde{q}^{Log}_{[t,T]} = Q_t \exp\left\{ \ln\left(\frac{\alpha_\tau^2}{\sqrt{2\beta_\tau}}\right) + \sqrt{\ln\left(\frac{2\beta_\tau}{\alpha_\tau^2}\right)} Z \right\}, \tag{3.68}$$

where α_τ and β_τ are given in Lemma 3.2.19.

(b) Reciprocal gamma (moment matching)

$$\tilde{q}^{IG}_{[t,T]} = Q_t \cdot IG(\alpha_{IG}(\tau), \beta_{IG}(\tau)), \tag{3.69}$$

where

$$\alpha_{IG}(\tau) = \frac{4\beta_\tau - \alpha_\tau^2}{2\beta_\tau - \alpha_\tau^2}, \qquad \beta_{IG}(\tau) = \frac{2\beta_\tau - \alpha_\tau^2}{2\alpha_\tau\beta_\tau},$$

and α_τ and β_τ are given in Lemma 3.2.19.

Remark:
Cumulative emissions in the model of Chesney and Taschini (2008) are given by (cf. Lemma 3.2.17)

$$\tilde{q}^{Lin}_{[t,T]} = Q_t \exp\left\{ \ln(\tau) + \left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}Z \right\}.$$

Proof :
(a): The parameters $\mu_L(t,T)$ and $\sigma_L(t,T)$ are chosen such that the first two moments of $\tilde{q}^{Log}_{[t,T]} = logN(\mu_L(t,T), \sigma_L^2(t,T))$ match those of $q_{[t,T]}$.
Hence by Lemma 3.2.19 and Lemma 3.2.24

$$\begin{aligned}
\sigma_L^2(t,T) &= \ln\left(\frac{m_2(t,T)}{m_1^2(t,T)}\right) = \ln\left(\frac{2Q_t^2\beta_{T-t}}{Q_t^2\alpha_{T-t}^2}\right) = \ln(2\beta_{T-t}) - 2\ln(\alpha_{T-t}), \\
\mu_L(t,T) &= \ln\left(m_1(t,T)\right) - \frac{1}{2}\sigma_L^2(t,T) = \ln(Q_t\alpha_{T-t}) - \frac{1}{2}\sigma_L^2(t,T) \\
&= \ln(Q_t) + 2\ln(\alpha_{T-t}) - \frac{1}{2}\ln(2\beta_{T-t}).
\end{aligned}$$

Note that $\sigma_L^2(t,T)$ is independent of Q_t, $\mu_L(t,T)$ not.
(b): The parameters α_{IG} and β_{IG} are chosen such that the first two moments of $\tilde{q}_{[t,T]}^{IG}$ match those of $q_{[t,T]}$. Hence by Lemma 3.2.19 and Lemma 3.2.26

$$\begin{aligned}
\alpha_{IG} &= \frac{2m_2(t,T) - m_1^2(t,T)}{m_2(t,T) - m_1^2(t,T)} = \frac{4Q_t^2\beta_\tau - Q_t^2\alpha_\tau^2}{2Q_t^2\beta_\tau - Q_t^2\alpha_\tau^2} = \frac{4\beta_\tau - \alpha_\tau^2}{2\beta_\tau - \alpha_\tau^2},\\
\beta_{IG} &= \frac{m_2(t,T) - m_1^2(t,T)}{m_1(t,T)m_2(t,T)} = \frac{2Q_t^2\beta_\tau - Q_t^2\alpha_\tau^2}{(Q_t\alpha_\tau)(2Q_t^2\beta_\tau)} = \frac{1}{Q_t}\cdot\frac{2\beta_\tau - \alpha_\tau^2}{2\alpha_\tau\beta_\tau}.
\end{aligned}$$

Equation (3.75) and (3.76) complete the proof. ♢

Permit pricing formulae

Lemma 3.2.22 (Permit price - Log-normal moment matching)
The permit price at time $t < T$ is given by

$$S_t^{Log} = \begin{cases} Pe^{-r\tau} & \text{if } q_{[0,t]} \geq N \\ Pe^{-r\tau}\cdot\Phi\left(\frac{-\ln\left(\frac{N-q_{[0,t]}}{Q_t}\right)+2\ln(\alpha_\tau)-\frac{1}{2}\ln(2\beta_\tau)}{\sqrt{\ln(2\beta_\tau)-2\ln(\alpha_\tau)}}\right) & \text{if } q_{[0,t]} < N \end{cases} \tag{3.70}$$

where $\tau = T - t$ is the time to compliance and α_τ, β_τ are given in Lemma 3.2.19.

Remark:
The permit price at time T is given by

$$S_T^{Lin} = P\cdot\mathbf{1}_{\{q_{[0,T]}\geq N\}}.$$

Proof :
Let $Z \sim N(0,1)$. Then,

$$S_t^{Log} \overset{(3.51),(3.68)}{=} \begin{cases} Pe^{-r\tau} & \text{if } q_{[0,t]} \geq N \\ Pe^{-r\tau}\cdot\mathbb{P}\left(\tilde{q}_{[t,T]}^{Log} > N - q_{[0,t]}|\mathcal{F}_t\right) & \text{if } q_{[0,t]} < N \end{cases}$$

where $\tilde{q}_{[t,T]}^{Log}$ is given in Lemma 3.2.21. Assuming that $N > q_{[0,t]}$ we have

$$\begin{aligned}
&\mathbb{P}\left(\tilde{q}_{[t,T]}^{Log} > N - q_{[0,t]}|\mathcal{F}_t\right)\\
&\quad = \mathbb{P}\left(Q_t\exp\left\{\ln\left(\frac{\alpha_\tau^2}{\sqrt{2\beta_\tau}}\right) + \sqrt{\ln\left(\frac{2\beta_\tau}{\alpha_\tau^2}\right)}Z\right\} > N - q_{[0,t]}|\mathcal{F}_t\right)
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{P}\left(Z > \frac{\ln\left(\frac{N-q_{[0,t]}}{Q_t}\right) - \ln\left(\frac{\alpha_\tau^2}{\sqrt{2\beta_\tau}}\right)}{\sqrt{\ln\left(\frac{2\beta_\tau}{\alpha_\tau^2}\right)}} | \mathcal{F}_t \right) \\
&= \Phi\left(\frac{-\ln\left(\frac{N-q_{[0,t]}}{Q_t}\right) + \ln\left(\frac{\alpha_\tau^2}{\sqrt{2\beta_\tau}}\right)}{\sqrt{\ln\left(\frac{2\beta_\tau}{\alpha_\tau^2}\right)}} \right) \\
&= \Phi\left(\frac{-\ln\left(\frac{N-q_{[0,t]}}{Q_t}\right) + 2\ln(\alpha_\tau) - \frac{1}{2}\ln(2\beta_\tau)}{\sqrt{\ln(2\beta_\tau) - 2\ln(\alpha_\tau)}} \right).
\end{aligned}
$$

$\diamondsuit$

Lemma 3.2.23 (Permit price - Reciprocal gamma moment matching)
The permit price at time $t < T$ *is given by*

$$
S_t^{IG} = \begin{cases} Pe^{-r\tau} & \text{if } q_{[0,t]} \geq N \\ Pe^{-r\tau} \cdot G\left(\frac{Q_t}{N-q_{[0,t]}} | \frac{4\beta_\tau - \alpha_\tau^2}{2\beta_\tau - \alpha_\tau^2}, \frac{2\beta_\tau - \alpha_\tau^2}{2\alpha_\tau\beta_\tau}\right) & \text{if } q_{[0,t]} < N \end{cases} \tag{3.71}
$$

where $\tau = T - t$ *is the time to compliance and* α_τ, β_τ *are given in Lemma 3.2.19.* $G(x|\alpha, \beta)$ *denotes the c.d.f. of a gamma distributed random variable with location parameter* α *and scale parameter* β *.*

Remark:
The permit price at time T is given by

$$
S_T^{Lin} = P \cdot \mathbf{1}_{\{q_{[0,T]} \geq N\}}.
$$

Proof :
By Equation (3.51) and (3.69)

$$
S_t^{IG} = \begin{cases} Pe^{-r\tau} & \text{if } q_{[0,t]} \geq N \\ Pe^{-r\tau} \cdot \mathbb{P}\left(\tilde{q}_{[t,T]}^{IG} > N - q_{[0,t]} | \mathcal{F}_t\right) & \text{if } q_{[0,t]} < N \end{cases}
$$

where $\tilde{q}_{[t,T]}^{IG} \sim IG(\alpha_{IG}(\tau), \beta_{IG}(\tau))$. $\alpha_{IG}(\tau)$ and $\beta_{IG}(\tau)$ are given in Lemma 3.2.21.

Thus for $N > q_{[0,t]}$ we have

$$\begin{aligned}
&\mathbb{P}\left(\tilde{q}^{IG}_{[t,T]} > N - q_{[0,t]}|\mathcal{F}_t\right) \\
&\quad = 1 - \mathbb{P}\left(\tilde{q}^{IG}_{[t,T]} \leq N - q_{[0,t]}|\mathcal{F}_t\right) \\
&\quad \overset{(3.76)}{=} 1 - \left(1 - G\left(\frac{Q_t}{N - q_{[0,t]}}|\alpha_{IG}(\tau), \beta_{IG}(\tau)\right)\right) \\
&\quad = G\left(\frac{Q_t}{N - q_{[0,t]}}|\alpha_{IG}(\tau), \beta_{IG}(\tau)\right) \\
&\quad = G\left(\frac{Q_t}{N - q_{[0,t]}}|\frac{4\beta_\tau - \alpha_\tau^2}{2\beta_\tau - \alpha_\tau^2}, \frac{2\beta_\tau - \alpha_\tau^2}{2\alpha_\tau\beta_\tau}\right).
\end{aligned}$$

$\diamondsuit$

Relevant distributions

Lemma 3.2.24 (Log-normal distribution)
Let $X \sim \log N(\mu, \sigma^2)$ *. Then*
(a) The kth moment of X *is given by*

$$\mu_k^{(X)} = e^{k\mu + k^2\frac{\sigma^2}{2}}.$$

(b) Let $X \sim logN(\mu, \sigma^2)$ *be a log-normal random variable with* $\mathbb{E}(X) = m_1$ *and* $\mathbb{E}(X^2) = m_2$ *. Then*

$$\sigma^2 = \ln\left(\frac{m_2}{m_1^2}\right), \qquad \mu = \ln(m_1) - \frac{1}{2}\sigma^2.$$

Proof :
(b) The first two moments of $X \sim \log N(\mu, \sigma^2)$ are given by

$$\begin{aligned}
\mathbb{E}(X) &= e^{\mu + \frac{\sigma^2}{2}}, \\
\mathbb{E}(X^2) &= e^{2(\mu+\sigma^2)} = (\mathbb{E}(X))^2 e^{\sigma^2}.
\end{aligned}$$

By assumption $m_2 = m_1^2 e^{\sigma^2}$ and $m_1 = e^{\mu}\sqrt{e^{\sigma^2}}$. $\diamondsuit$

Lemma 3.2.25 (Gamma and Reciprocal Gamma distribution)
(a) The probability density function (p.d.f.) of a Gamma distributed random variable with shape parameter α *and scale parameter* β *is*

$$g(x \mid \alpha, \beta) = \frac{1}{\Gamma(\alpha)} \beta^{-\alpha} x^{\alpha-1} \exp\left\{-\frac{x}{\beta}\right\} \;\; \forall x > 0. \tag{3.72}$$

The p.d.f. and the cumulative distribution function (c.d.f.) of $X \sim \Gamma(\alpha, \beta)$ *, denoted by* $g(x \mid \alpha, \beta)$ *and* $G(x \mid \alpha, \beta)$ *, respectively, have the following properties*

$$g(x \mid \alpha, \beta) = \frac{x}{\beta(\alpha-1)} g\left(x \mid \alpha - 1, \beta\right) \;\; \forall \alpha > 1, \tag{3.73}$$

$$g(x \mid \alpha, \beta) = \frac{1}{\beta} g\left(\frac{x}{\beta} \mid \alpha, 1\right), \tag{3.74}$$

$$G(x \mid \alpha, \beta) = G\left(\frac{x}{\beta} \mid \alpha, 1\right). \tag{3.75}$$

(b) If $X \sim \Gamma(\alpha, \beta)$ *then* $\frac{1}{X} \sim IG(\alpha, \beta)$ *.*
Denote the c.d.f. and p.d.f. of $\frac{1}{X} \sim IG(\alpha, \beta)$ *by* $G_R(x \mid \alpha, \beta)$ *and* $g_R(x \mid \alpha, \beta)$ *, respectively. They are related to* $G(\cdot)$ *and* $g(\cdot)$ *as follows:*

$$G_R(x \mid \alpha, \beta) = 1 - G\left(\frac{1}{x} \mid \alpha, \beta\right), \tag{3.76}$$

$$g_R(x \mid \alpha, \beta) = \frac{1}{x^2} g\left(\frac{1}{x} \mid \alpha, \beta\right) \forall x > 0. \tag{3.77}$$

Hence the p.d.f. of $\frac{1}{X}$ *is*

$$g_R(x \mid \alpha, \beta) = \frac{1}{\Gamma(\alpha)} \beta^{-\alpha} x^{-\alpha-1} \exp\left\{-\frac{1}{\beta x}\right\} \;\; \forall x > 0. \tag{3.78}$$

Lemma 3.2.26 (Moments of reciprocal gamma distribution)
The first two moments of $X \sim IG(\alpha, \beta)$ *are*

$$M_1 = \frac{1}{\beta(\alpha-1)}, \qquad M_2 = \frac{1}{\beta^2(\alpha-1)(\alpha-2)}. \tag{3.79}$$

and the parameters α *and* β *can be expressed in terms of the first two moments*

$$\alpha = \frac{2M_2 - M_1^2}{M_2 - M_1^2}, \qquad \beta = \frac{M_2 - M_1^2}{M_1 M_2}. \tag{3.80}$$

3.3 Reduced-form models

3.3a Reduced-form model of Carmona et al. (2009a)

Motivation

Reduced-form models have been introduced by Carmona et al. (2009a) in a recent paper that addresses the problem of pricing option contracts on emission permits. Following the definition of Carmona et al. (2009a), we call simplified equilibrium models reduced-form models.

The basic assumption in the model of Carmona et al. (2009a) is that the permit price is described by the following risk-neutral price dynamics, i.e. under the risk-neutral measure $\mathbb{Q}$, the futures permit price $F(t,T)$ is modelled as

$$F(t,T) = P \cdot \mathbb{Q}\left(\Gamma_T > 1 \mid \mathcal{F}_t\right) = P \cdot \mathbb{Q}\left(\Gamma_0 \exp\left\{\int_0^T \sigma_s dW_s - \frac{1}{2}\int_0^T \sigma_s^2 ds\right\} > 1 \mid \mathcal{F}_t\right), \tag{3.81}$$

where $\Gamma_0 \in (0,\infty)$, and $\sigma(\cdot)$ is a continuous square integrable deterministic function. Carmona et al. (2009a) do not provide an explanation of the relationship between their reduced-form model and the existing stochastic equilibrium models. Probably, the main motivation of their definition is that those dynamics yield a tractable option pricing model and that the dynamics take the specific permit price characteristics into account.

Carmona et al. (2009a) derive an explicit permit pricing formula for their reduced-form model. In order to simplify the notation the formula is derived for the normalized permit prices as given in Definition 3.3.1. Based on the permit pricing formula, a stochastic differential equation (SDE) for the normalized permit price is derived (cf. Lemma 3.3.2). Parameter estimation of the reduced-form model can be carried out with the help of this SDE (cf. Section 4.3).

Definition 3.3.1 (Normalized permit price)
Let $F(t,T)$ be the permit price given in Equation (3.50) and let P be the penalty. Then the permit price divided by the penalty is defined as

$$a_t = \frac{F(t,T)}{P}.$$

Permit price formulae

Lemma 3.3.2 (Permit price in the model of Carmona et al. (2009a))
Assume that the permit price is given by Equation (3.81) for $\Gamma_0 \in (0, \infty)$ *and a deterministic function* σ_s *.*
Then under the risk-neutral measure the permit price is given by

$$a_t = \Phi\left(\frac{\Phi^{-1}(a_0)\sqrt{\int_0^T \sigma_s^2 ds} + \int_0^t \sigma_s dW_s}{\sqrt{\int_t^T \sigma_s^2 ds}}\right). \tag{3.82}$$

Proof :

$$\begin{aligned}
a_t &\overset{(3.81)}{=} \mathbb{Q}\left(\Gamma_T > 1 \mid \mathcal{F}_t\right) = \mathbb{Q}\left(\Gamma_t \exp\left\{\int_t^T \sigma_s dW_s - \frac{1}{2}\int_t^T \sigma_s^2 ds\right\} > 1 \mid \mathcal{F}_t\right) \\
&= \Phi\left(\frac{\ln(\Gamma_t) - \frac{1}{2}\int_t^T \sigma_s^2 ds}{\sqrt{\int_t^T \sigma_s^2 ds}}\right) \\
&= \Phi\left(\frac{\ln(\Gamma_0) + \int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^T \sigma_s^2 ds}{\sqrt{\int_t^T \sigma_s^2 ds}}\right) \\
&= \Phi\left(\frac{\ln(\Gamma_0) - \frac{1}{2}\int_0^T \sigma_s^2 ds}{\sqrt{\int_t^T \sigma_s^2 ds}} + \frac{\int_0^t \sigma_s dW_s}{\sqrt{\int_t^T \sigma_s^2 ds}}\right) \\
&= \Phi\left(\frac{\ln(\Gamma_0) - \frac{1}{2}\int_0^T \sigma_s^2 ds}{\sqrt{\int_t^T \sigma_s^2 ds}} \cdot \frac{\sqrt{\int_0^T \sigma_s^2 ds}}{\sqrt{\int_0^T \sigma_s^2 ds}} + \frac{\int_0^t \sigma_s dW_s}{\sqrt{\int_t^T \sigma_s^2 ds}}\right) \\
&= \Phi\left(\frac{\ln(\Gamma_0) - \frac{1}{2}\int_0^T \sigma_s^2 ds}{\sqrt{\int_0^T \sigma_s^2 ds}} \cdot \frac{\sqrt{\int_0^T \sigma_s^2 ds}}{\sqrt{\int_t^T \sigma_s^2 ds}} + \frac{\int_0^t \sigma_s dW_s}{\sqrt{\int_t^T \sigma_s^2 ds}}\right) \\
&= \Phi\left(\Phi^{-1}(a_0) \cdot \frac{\sqrt{\int_0^T \sigma_s^2 ds}}{\sqrt{\int_t^T \sigma_s^2 ds}} + \frac{\int_0^t \sigma_s dW_s}{\sqrt{\int_t^T \sigma_s^2 ds}}\right).
\end{aligned}$$

$\diamondsuit$

Lemma 3.3.3 (General SDE for the model of Carmona et al. (2009a))
The dynamics of the permit price under the historical measure are given by

$$da_t = \Phi'(\Phi^{-1}(a_t))\sqrt{\frac{\sigma_t^2}{\int_t^T \sigma_s^2 ds}}(dW_t + hdt). \tag{3.83}$$

Proof :
First, the dynamics of the permit price under the risk-neutral measure are derived:
By Lemma 3.3.2,

$$a_t = \Phi\left(\frac{\Phi^{-1}(a_0)\sqrt{\int_0^T \sigma_s^2 ds} + \int_0^t \sigma_s dW_s}{\sqrt{\int_t^T \sigma_s^2 ds}}\right) := \Phi\left(X_t\right) := \Phi\left(\frac{z_t}{\sqrt{n_t}}\right).$$

Applying Itô's lemma yields

$$\begin{aligned} da_t = &\ d\Phi\left(X_t\right) = \Phi'(X_t)dX_t + \frac{1}{2}\Phi''(X_t)d[X]_t \\ = &\ \Phi'(X_t)dX_t - \frac{1}{2}X_t\Phi'(X_t)d[X]_t \\ = &\ \Phi'(X_t)\left[dX_t - \frac{1}{2}X_t d[X]_t\right], \end{aligned}$$

where

$$\begin{aligned} dn_t = &\ -\sigma_t^2 dt, \\ dz_t = &\ \sigma_t dW_t, \\ dX_t = &\ \frac{1}{\sqrt{n_t}}dz_t - \frac{1}{2}\frac{X_t}{n_t}dn_t = \frac{1}{\sqrt{n_t}}\sigma_t dW_t + \frac{1}{2}\frac{X_t}{n_t}\sigma_t^2 dt, \\ d[X]_t = &\ \frac{\sigma_t^2}{n_t}dt. \end{aligned}$$

Thus

$$dX_t - \frac{1}{2}X_t d[X]_t = \frac{1}{\sqrt{n_t}}\sigma_t dW_t + \frac{1}{2}\frac{X_t}{n_t}\sigma_t^2 dt - \frac{1}{2}X_t\frac{\sigma_t^2}{n_t}dt = \frac{1}{\sqrt{n_t}}\sigma_t dW_t.$$

Therefore, the permit price dynamics under the risk-neutral measure are given by

$$da_t = \Phi'(\Phi^{-1}(a_t))\sqrt{\frac{\sigma_t^2}{\int_t^T \sigma_s^2 ds}}dW_t.$$

Assuming that the market price of risk process is constant and deterministic with value h, Carmona et al. (2009a) show that the dynamics under the historical measure are given by Equation (3.83). ♢

Remark:
Even though, Carmona et al. (2009a) derive an SDE of the reduced-form model for an arbitrary function σ_s , they only use a specific function σ_s for pricing options (cf. Corollary 3.3.4).

Corollary 3.3.4 (SDE for reduced-form model of Carmona et al. (2009a))
Under the historical measure the permit price dynamics for

$$\sigma_t^2 = \beta(T-t)^{\beta-1}T^{-\beta}$$

are given by

$$\begin{aligned} da_t &= \Phi'(\Phi^{-1}(a_t))\sqrt{\frac{\beta}{T-t}}(dW_t + hdt) && (3.84)\\ &= \Phi'(\Phi^{-1}(a_t))\left[\sqrt{\frac{\beta}{T-t}}dW_t + h\sqrt{\frac{\beta}{T-t}}dt\right]. && (3.85)\end{aligned}$$

Proof :
Follows directly from Lemma 3.3.3. ♢

3.3b Reduced-form model of Grüll and Taschini (2009)

A by-product of deriving a parameter estimation method for the stochastic equilibrium models of Chesney and Taschini (2008) and Grüll and Kiesel (2009) is the definition of the following reduced-form model. Readers interested in the motivation of the SDE given in Lemma 3.3.5 are referred to the proof in Section 4.3.

Definition 3.3.5 (Reduced-form model of Grüll and Taschini)
Assume that the permit price divided by the penalty is described by the following SDE

$$d\left(\Phi^{-1}(a_t)\sqrt{T-t}\right) = adt + bdW_t,$$

where $a, b \in \mathbb{R}$ are the parameters of the reduced-form model under the historical measure.

In the following Corollary we derive an SDE for the reduced-form model of Grüll and Taschini (2009) in order to compare it to the model of Carmona et al. (2009a).

Corollary 3.3.6 (SDE for the reduced-form model of Grüll and Taschini)
The permit price dynamics are given by

$$da_t = \frac{\Phi'(\Phi^{-1}(a_t))}{\sqrt{T-t}}\left[\left(a + \frac{1-b^2}{2\sqrt{T-t}}\Phi^{-1}(a_t)\right)dt + bdW_t\right].$$

Proof :
Let $X_t = \Phi^{-1}(a_t)\sqrt{T-t}$. Thus $a_t = \Phi\left(\frac{X_t}{\sqrt{T-t}}\right) := f(X_t, t)$ and

$$\begin{aligned}
f_x(x,t) &= \Phi'\left(\frac{x}{\sqrt{T-t}}\right) \cdot \frac{1}{\sqrt{T-t}}, \\
f_{xx}(x,t) &= \Phi''\left(\frac{x}{\sqrt{T-t}}\right) \cdot \frac{1}{T-t} = -\frac{x}{\sqrt{T-t}}\Phi'\left(\frac{x}{\sqrt{T-t}}\right) \cdot \frac{1}{T-t}, \\
f_t(x,t) &= \frac{1}{2}x\Phi'\left(\frac{x}{\sqrt{T-t}}\right) \cdot (T-t)^{-\frac{3}{2}}.
\end{aligned}$$

By Definition 3.3.5, we have

$$\begin{aligned}
dX_t &= adt + bdW_t, \\
d[X]_t &= b^2 dt.
\end{aligned}$$

By Itô's lemma, we obtain

$$\begin{aligned}
da_t = &\ df(X_t,t) = f_x(X_t,t)dX_t + f_t(X_t,t)dt + \frac{1}{2}f_{xx}(X_t,t)d[X]_t \\
= &\ \Phi'\left(\frac{X_t}{\sqrt{T-t}}\right) \cdot \frac{1}{\sqrt{T-t}}\left[adt + bdW_t + \frac{X_t}{2(T-t)}dt - \frac{X_t}{2(T-t)} \cdot b^2 dt\right] \\
= &\ \frac{\Phi'(\Phi^{-1}(a_t))}{\sqrt{T-t}}\left[\left(a + \frac{1-b^2}{2\sqrt{T-t}}\Phi^{-1}(a_t)\right)dt + bdW_t\right].
\end{aligned}$$

◊

Remark:

The SDE for the reduced-form model of Grüll and Taschini (2009) differs from the SDE for the model of Carmona et al. (2009a) by the additional term $\frac{1-b^2}{2\sqrt{T-t}}\Phi^{-1}(a_t)dt$.

3.4 Relationship between deterministic and stochastic equilibrium models

Deterministic and stochastic equilibrium models are developed for analyzing permit prices and properties of emissions trading schemes from a theoretical point of view. All the models in Section 3.1 and 3.2 derive permit prices and prove cost-optimality of emissions trading with the help of optimization problems. The basic assumption is that regulated companies are maximizing their profits by choosing optimal strategies of emitting greenhouse gases and buying or selling permits. Moreover, in the joint optimization problem a fictitious central planner is maximizing aggregated profits of all firms by choosing optimal emission quantities for each firm. The most important variables of the optimization problems are deterministic/stochastic in a deterministic/stochastic equilibrium model. However, the structure of the optimization problems is similar for all the discussed models (cf. Section 3.1 and 3.2). Readers interested in a detailed comparison are referred to Table 3.2 - 3.7.

A by-product of solving the profit maximization problems in the different settings is that we obtain a convenient interpretation of the permit price. In a deterministic equilibrium model the permit price is equal to the marginal abatement costs (cf. Section 3.1) whereas in a stochastic equilibrium model the permit price is equal to the penalty multiplied by the probability of a permit shortage at the end of the compliance period (cf. Section 3.2). In a deterministic framework the permit price (i.e. the marginal abatement costs) does neither explicitly depend on the regulations of an emissions trading scheme such as the penalty fee and the number of allocated permits nor on the expected future emissions of the regulated companies. Permit prices in a stochastic equilibrium model capture these dependencies. At a fixed point in time the probability of permit shortage can be interpreted as the marginal abatement costs that depend on the expected cumulative emissions in the compliance period and on the total number of permits functional for compliance. This interpretation is the link between the two concepts: the concept of marginal abatement costs and the concept of probability of permit shortage.

Paper	**Single- or Multi-step model**	**Discrete- or Continuous-time**	**Deterministic or Stochastic variables**	**Contains proof of cost optimality**
Montgomery (1972)	Single	Discrete	Deterministic	Yes
Cronshaw and Kruse (1996)	Multi	Discrete	Deterministic	Yes
Rubin (1996)	Multi	Continuous	Deterministic	Yes
Kling and Rubin (1997)	Multi	Continuous	Deterministic	No[1]
Seifert et al. (2008)	Multi	Continuous	Stochastic	Yes
Carmona et al. (2009)	Multi	Discrete	Stochastic	Yes
Chesney and Taschini (2008)	Multi	Continuous	Stochastic	No
Grüll and Kiesel (2009)	Multi	Continuous	Stochastic	No

Table 3.2: Survey on the general framework of deterministic and stochastic equilibrium models.

[1] No, but contains proof of social optimality of emissions trading.

Model	**Costs/Profits in the target function of the optimization problem**	
Montgomery (1972)	(i)	Abatement costs defined as the difference between the maximum unconstrained profits from the production of goods and the maximum profits when the firms have to adopt to a specific emission level
	(ii)	Costs/Profits from emissions trading
Rubin (1996)	(i)	Abatement costs as defined by Montgomery (1972)
	(ii)	Costs/Profits from emissions trading
Kling and Rubin (1997)	(i)	Revenues from the production of goods
	(ii)	Production costs depending on the number of produced goods and on the emissions
	(iii)	Costs/Profits from emissions trading
Cronshaw and Kruse (1996)	(i)	Maximum profits from the production of goods depending on the emissions and on the number of traded permits
	(ii)	Costs/Profits from emissions trading
Seifert et al. (2008)	(i)	Abatement costs
	(ii)	Costs/Profits from emissions trading
	(iii)	Penalty payment
Carmona et al. (2009b)	(i)	Revenues from the production of goods depending on the output volume and the production technology
	(ii)	Production costs depending on the output volume and on the production technology
	(iii)	Costs/Profits from emissions trading
	(iv)	Penalty payment

Table 3.3: Comparison of the optimization problems in the equilibrium models (Part 1).

Model	**Cost optimization with respect to the number of traded permits and**	**Constraints of the optimization problem**
Montgomery (1972)	Total emissions	Firms comply at the end
Rubin (1996)	Total emissions	Firms comply at the end and are not allowed to borrow permits during the compliance period
Kling and Rubin (1997)	Total emissions and Output of produced good	Firms comply at the end
Cronshaw and Kruse (1996)	Total emissions	Firms comply at the end and are not allowed to borrow permits during the compliance period
Seifert et al. (2008)	Abated emissions	None
Carmona et al. (2009b)	Output of produced goods using different technologies	None

Table 3.4: Comparison of the optimization problems in the equilibrium models (Part 2).

Model	**Literature used to prove cost-optimality of emissions trading**
Montgomery (1972)	Well-known Karush-Kuhn-Tucker conditions - theorem was originally derived by Karush (1939) and Kuhn and Tucker (1951)
Rubin (1996)	Barro and Sala-i-Martin (1995) or Kamien and Schwartz (1991), Seierstad and Sydsaeter (1987), Steinberg and Stalford (1973)
Kling and Rubin (1997)	Kamien and Schwartz (1991)
Cronshaw and Kruse (1996)	Takayama (1985)
Seifert et al. (2008)	Sethi and Thompson (1981)
Carmona et al. (2009b)	Tailor-made proof by Carmona et al. (2009b)
Chesney and Taschini (2008)	Reference to the proof in Carmona et al. (2009b)
Grüll and Kiesel (2009)	Reference to the proof in Carmona et al. (2009b)

Table 3.5: Overview of the literature used to prove cost-optimality of emissions trading in the different equilibrium models.

Variable	Description
α	Abatement rate in the model of Seifert et al. (2008)
A_t	Time-t futures permit price (maturity at time T) in the model of Carmona et al. (2009b)
B	Number of permits in the "bank" corresponding to the number of allocated permits plus the purchased permits minus the emissions. Variable is used in the models of Rubin (1996), Kling and Rubin (1997) and Cronshaw and Kruse (1996)
β	Emission rate before abatement activities in the model of Seifert et al. (2008)
$C(\cdot)$	Abatement costs in the models of Montgomery (1972) and Rubin (1996)
$C_{\text{good}}(\cdot)$	Production costs in the models of Montgomery (1972), Rubin (1996), Kling and Rubin (1997), Cronshaw and Kruse (1996) and Carmona et al. (2009b)
D	Demand for the goods in the model of Carmona et al. (2009b)
Δ	Aggregated uncontrollable emissions in $[0, T]$ in the model of Carmona et al. (2009b)
e	Emission factor in the model of Carmona et al. (2009b)
G	Prices of the produced goods in the model of Montgomery (1972), Rubin (1996), Kling and Rubin (1997) and Carmona et al. (2009b)
i	Superscript i is always referring to company i
K	Production capacity in the model of Carmona et al. (2009b)
κ	Marginal production costs in the model of Carmona et al. (2009b)
μ	Parameter of the process modelling the emission rate in the models of Chesney and Taschini (2008) and Grüll and Kiesel (2009)
N	Number of allocated emission allowances in the models of Montgomery (1972), Rubin (1996), Kling and Rubin (1997), Cronshaw and Kruse (1996), Seifert et al. (2008), Carmona et al. (2009b), Chesney and Taschini (2008) and Grüll and Kiesel (2009)
P	Penalty per unit of emission that is not covered by an allowance at compliance time. Variable is used in the models of Seifert et al. (2008), Carmona et al. (2009b), Chesney and Taschini (2008) and Grüll and Kiesel (2009)

Table 3.6: Survey on the variables of the different equilibrium models (Part 1).

Variable	Description
$\pi(\cdot)$	Profit/Loss from the production of goods in the models of Montgomery (1972) and Rubin (1996)
$\Pi(\cdot)$	Maximum profit from the production of goods in the model of Cronshaw and Kruse (1996)
Q	Emission rate (including abatement activities) in the models of Montgomery (1972), Rubin (1996), Kling and Rubin (1997), Cronshaw and Kruse (1996), Chesney and Taschini (2008) and Grüll and Kiesel (2009)
q	Expected total cumulative emissions in $[0, T]$ in the model of Seifert et al. (2008)
$q(y)$	Cumulative emissions in $[0, T]$ (excluding uncontrollable emissions) in the model Carmona et al. (2009b)
$q_{[t_1,t_2]}$	Total cumulative emissions in $[t_1, t_2]$ in the models of Chesney and Taschini (2008) and Grüll and Kiesel (2009)
$R_{\text{good}}(\cdot)$	Revenues from the production of goods in the models of Montgomery (1972), Rubin (1996), Kling and Rubin (1997), Cronshaw and Kruse (1996) and Carmona et al. (2009b)
S	Permit price in the models of Montgomery (1972), Rubin (1996), Kling and Rubin (1997), Cronshaw and Kruse (1996), Seifert et al. (2008), Chesney and Taschini (2008) and Grüll and Kiesel (2009)
σ	Parameter of the process modelling the emission rate in the models of Chesney and Taschini (2008) and Grüll and Kiesel (2009)
T	End of the (compliance) period in the models of Rubin (1996), Kling and Rubin (1997), Cronshaw and Kruse (1996), Seifert et al. (2008), Carmona et al. (2009b), Chesney and Taschini (2008) and Grüll and Kiesel (2009)
θ	Number of permits bought from/ sold to other companies at time t in the models of Montgomery (1972), Rubin (1996), Kling and Rubin (1997), Cronshaw and Kruse (1996) and Seifert et al. (2008)
Θ	Number of permits bought from/ sold to other companies until time t in the model of Carmona et al. (2009b)
y	Output quantity of the produced good in the models of Montgomery (1972), Rubin (1996), Kling and Rubin (1997), Cronshaw and Kruse (1996) and Carmona et al. (2009b)

Table 3.7: Survey on the variables of the different equilibrium models (Part 2).

3.5 Relationship between stochastic equilibrium models and reduced-form models

This section discusses how the reduced-form model of Carmona et al. (2009a) is related with the stochastic equilibrium model of Carmona et al. (2009b) and with the extended stochastic equilibrium model of Chesney and Taschini (2008).

Relationship between the reduced-form model of Carmona et al. (2009a) and the stochastic equilibrium model of Carmona et al. (2009b)

The basic assumption in the reduced-form model of Carmona et al. (2009a) is that the permit price is described by the following risk-neutral price dynamics, i.e. under the risk-neutral measure $\mathbb{Q}$, the futures permit price $F(t,T)$ is modelled as

$$F(t,T) = P \cdot \mathbb{Q}\left(\Gamma_T > 1 \mid \mathcal{F}_t\right) = P \cdot \mathbb{Q}\left(\Gamma_0 \exp\left\{\int_0^T \sigma_s dW_s - \frac{1}{2}\int_0^T \sigma_s^2 ds\right\} > 1 \mid \mathcal{F}_t\right),$$

where $\Gamma_0 \in (0,\infty)$, and $\sigma(\cdot)$ is a continuous square integrable deterministic function. Carmona et al. (2009a) prove that the futures permit price under the historical measure $\mathbb{P}$ for some fixed $h \in \mathbb{R}$, is given by

$$\begin{aligned} F(t,T) &= P \cdot \mathbb{P}\left(\Gamma_0 \exp\left\{\int_0^T \sigma_s (dW_s + h ds) - \frac{1}{2}\int_0^T \sigma_s^2 ds\right\} > 1 \mid \mathcal{F}_t\right) \\ &= P \cdot \mathbb{P}\left(\Gamma_0 \exp\left\{\int_0^T \sigma_s dW_s - \frac{1}{2}\int_0^T (\sigma_s^2 - 2h\sigma_s) ds\right\} > 1 \mid \mathcal{F}_t\right). \end{aligned} \tag{3.86}$$

Comparing Equation (3.50) and (3.86), shows that the reduced-form model of Carmona et al. (2009a) is a stochastic equilibrium model with aggregated cumulative emissions as given in Lemma 3.5.1.

Lemma 3.5.1 (Cumulative emissions in the model of Carmona et al. (2009a))
Let $\Gamma_0 \in (0,\infty)$, $h \in \mathbb{R}$. $\sigma(s)$ *denotes a continuous square integrable deterministic function. Then the cumulative emissions in the interval* $[0,T]$ *are given by*

$$q_{[0,T]} = N \cdot \Gamma_0 \exp\left\{\int_0^T \sigma(s) dW_s - \frac{1}{2}\int_0^T (\sigma^2(s) - 2h\sigma(s)) ds\right\}. \tag{3.87}$$

Remark:
It is important to note that the cumulative emissions of the reduced-form model of Carmona et al. (2009a) do not satisfy two important (and quite natural) properties of pollutants such as greenhouse gases. In the reduced-form model cumulative emissions are

- not additive in time (i.e. $q_{[0,T]} \neq q_{[0,t]} + q_{[t,T]}$ for $t < T$) and
- they do not strictly increase over time.

However, this assumption makes computations in Carmona et al. (2009a) much easier and yields a tractable option pricing model.

Relationship between the reduced-form model of Carmona et al. (2009a) and the stochastic equilibrium model of Chesney and Taschini (2008)

The comparison of the models of Carmona et al. (2009a) and the extended model of Chesney and Taschini (2008) is performed by deriving a stochastic differential equation (SDE) for the extended model of Chesney and Taschini (2008) (cf. Lemma 3.5.4) and then comparing it to the SDE of the reduced-form model of Carmona et al. (2009a) (cf. Corollary 3.3.4).

Analogous to the proof in the paper of Carmona et al. (2009a), the derivation of the price dynamics is done in two steps. First, we derive the theoretical price of emission permits at time t in the extended framework of Chesney and Taschini (2008), assuming that we know the emission rate at time t and the aggregated cumulative emissions until t (cf. Lemma 3.5.2). The SDE for the price dynamics is obtained in a second step by treating the emission rate and cumulative emissions in Lemma 3.5.2 as random variables.

Lemma 3.5.2 (Normalized permit price in the extended model of CT)
The time-t permit price divided by the penalty is given by

$$a_t = \Phi\left(\frac{-\ln\left(\frac{N-q_{[0,t]}}{\tau\cdot Q_t}\right) + \int_t^T \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right) ds}{\sqrt{\int_t^T \sigma^2(s) ds}}\right).$$

In particular, we have

$$a_0 = \Phi\left(\frac{-\ln\left(\frac{N}{\tau\cdot Q_0}\right) + \int_0^T \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right) ds}{\sqrt{\int_0^T \sigma^2(s) ds}}\right).$$

Proof :
Follows directly from $a_t = \frac{F(t,T)}{P}$ and Lemma 3.2.18. $\diamondsuit$

Definition 3.5.3 ("Longness" of the permit market)
The "longness" of the permit market is defined as the number of remaining permits divided by the emissions in the remaining time period given the current emission rate, i.e.

$$\frac{N - q_{[0,t]}}{(T-t) \cdot Q_t}.$$

Remark:
The "longness" of the permit market has to be interpreted as follows: Values greater (less) than 1 correspond to a situation where the emission market is long (short) in permits.

Theorem 3.5.4 (SDE for the extended model of Chesney and Taschini)
Assume that the emission rate follows a geometric Brownian motion with a deterministic time-dependent drift μ_s and a diffusion coefficient σ_s .

(a) Approximate the "longness" of the permit market by

$$\frac{N - q_{[0,t]}}{(T-t) \cdot Q_t} \approx \frac{N}{T \cdot Q_0} \exp\left\{ \int_0^t \left(\tilde{\mu}_s - \frac{\tilde{\sigma}_s^2}{2} \right) ds + \int_0^t \tilde{\sigma}_s dW_s \right\},$$

where $\tilde{\mu}_s$ and $\tilde{\sigma}_s$ are deterministic functions. Then the dynamics of the permit price in the extended model of Chesney and Taschini (2008) are given by

$$da_t = -\frac{\Phi'\left(\Phi^{-1}(a_t)\right)}{\sqrt{\int_t^T \sigma_s^2 ds}} \left[\left(\tilde{\mu}_t + \mu_t - \frac{\tilde{\sigma}_t^2}{2} - \frac{\sigma_t^2}{2} + \frac{1}{2} \frac{\tilde{\sigma}_t^2 - \sigma_t^2}{\sqrt{\int_t^T \sigma_s^2 ds}} \Phi^{-1}(a_t) \right) dt + \tilde{\sigma}_t dW_t \right].$$

(b) The model of Chesney and Taschini (2008) with time-dependent emission rate can be transformed into the model of Carmona et al. (2009a) as given in Corollary 3.3.4 by setting

$$\begin{aligned} \tilde{\sigma}_t =& -\sigma_t = -\sqrt{\beta(T-t)^{\beta-1}}, \\ \tilde{\mu}_t =& -\mu_t + \sigma_t(\sigma_t - h). \end{aligned}$$

Proof :
(a) The assumed approximation and Lemma 3.5.2 yield

$$a_t = \Phi\left(\frac{-\ln\left(\frac{N}{T \cdot Q_0} \exp\left\{ \int_0^t \left(\tilde{\mu}_s - \frac{\tilde{\sigma}_s^2}{2} \right) ds + \int_0^t \tilde{\sigma}_s dW_s \right\} \right) + \int_t^T \left(\mu_s - \frac{\sigma_s^2}{2} \right) ds}{\sqrt{\int_t^T \sigma_s^2 ds}} \right)$$

$$
\begin{aligned}
&= \Phi\left(\frac{-\ln\left(\frac{N}{T\cdot Q_0}\right)-\int_0^t\left(\tilde{\mu}_s-\frac{\tilde{\sigma}_s^2}{2}\right)ds-\int_0^t\tilde{\sigma}_s dW_s+\int_t^T\left(\mu_s-\frac{\sigma_s^2}{2}\right)ds}{\sqrt{\int_t^T\sigma_s^2 ds}}\right)\\
&= \Phi\left(\frac{-\ln\left(\frac{N}{T\cdot Q_0}\right)-\int_0^t\left(\tilde{\mu}_s-\frac{\tilde{\sigma}_s^2}{2}\right)ds-\int_0^t\tilde{\sigma}_s dW_s+\int_0^T\left(\mu_s-\frac{\sigma_s^2}{2}\right)ds-\int_0^t\left(\mu_s-\frac{\sigma_s^2}{2}\right)ds}{\sqrt{\int_t^T\sigma_s^2 ds}}\right)\\
&= \Phi\left(\frac{\Phi^{-1}(a_0)\sqrt{\int_0^T\sigma_s^2 ds}-\int_0^t\left(\tilde{\mu}_s+\mu_s-\frac{\tilde{\sigma}_s^2}{2}-\frac{\sigma_s^2}{2}\right)ds-\int_0^t\tilde{\sigma}_s dW_s}{\sqrt{\int_t^T\sigma_s^2 ds}}\right)\\
&:= \Phi\left(X_t\right) := \Phi\left(\frac{z_t}{\sqrt{n_t}}\right).
\end{aligned}
$$

We have

$$
\begin{aligned}
da_t =&\ d\Phi\left(X_t\right) = \Phi'(X_t)dX_t + \frac{1}{2}\Phi''(X_t)d[X]_t\\
=&\ \Phi'(X_t)dX_t - \frac{1}{2}X_t\Phi'(X_t)d[X]_t\\
=&\ \Phi'(X_t)\left[dX_t - \frac{1}{2}X_t d[X]_t\right],
\end{aligned}
$$

where

$$
\begin{aligned}
dn_t =&\ -\sigma_t^2 dt,\\
dz_t =&\ -\left(\tilde{\mu}_t+\mu_t-\frac{\tilde{\sigma}_t^2}{2}-\frac{\sigma_t^2}{2}\right)dt-\tilde{\sigma}_t dW_t,\\
dX_t =&\ \frac{1}{\sqrt{n_t}}dz_t-\frac{1}{2}\frac{X_t}{n_t}dn_t\\
=&\ -\frac{1}{\sqrt{\int_t^T\sigma_s^2 ds}}\left(\tilde{\mu}_t+\mu_t-\frac{\tilde{\sigma}_t^2}{2}-\frac{\sigma_t^2}{2}\right)dt-\frac{\tilde{\sigma}_t}{\sqrt{\int_t^T\sigma_s^2 ds}}dW_t+\frac{1}{2}\frac{\sigma_t^2}{\int_t^T\sigma_s^2 ds}X_t dt,\\
d[X]_t =&\ \frac{\tilde{\sigma}_t^2}{\int_t^T\sigma_s^2 ds}dt.
\end{aligned}
$$

Thus

$$
\begin{aligned}
dX_t-\frac{1}{2}X_t d[X]_t =&\ -\frac{1}{\sqrt{\int_t^T\sigma_s^2 ds}}\left(\tilde{\mu}_t+\mu_t-\frac{\tilde{\sigma}_t^2}{2}-\frac{\sigma_t^2}{2}\right)dt-\frac{\tilde{\sigma}_t}{\sqrt{\int_t^T\sigma_s^2 ds}}dW_t\\
&-\frac{1}{2}\frac{\tilde{\sigma}_t^2-\sigma_t^2}{\int_t^T\sigma_s^2 ds}X_t dt.
\end{aligned}
$$

(b) The model of Chesney and Taschini (2008) can be transformed into the model of Carmona et al. (2009a) equating the coefficients of “ dt ” and “ dW_t ”

$$
-\frac{\tilde{\sigma}_t}{\sqrt{\int_t^T\sigma_s^2 ds}} = \sqrt{\frac{\beta}{T-t}}, \tag{3.88}
$$

and

$$-\frac{1}{\sqrt{\int_t^T \sigma_s^2 ds}}\left(\tilde{\mu}_t + \mu_t - \frac{\tilde{\sigma}_t^2}{2} - \frac{\sigma_t^2}{2}\right) = h\sqrt{\frac{\beta}{T-t}}. \tag{3.89}$$

By setting $\tilde{\sigma}_t^2 = \sigma_t^2$ and then rearranging Equation (3.88) we obtain the following PDE

$$\tilde{\sigma}_t^2 \cdot \frac{T-t}{\beta} = \int_t^T \tilde{\sigma}_s^2 ds.$$

Hence for $\beta > 0$ we have $\tilde{\sigma}_t^2 = \beta(T-t)^{\beta-1} = \sigma_t^2$. Thus

$$\tilde{\sigma}_t = -\sigma_t = -\sqrt{\beta(T-t)^{\beta-1}}.$$

Applying $\tilde{\sigma}_t^2 = \sigma_t^2$ to Equation (3.89) and then equating (3.88) and (3.89) yields

$$\tilde{\mu}_t + \mu_t - \tilde{\sigma}_t^2 = h\tilde{\sigma}_t$$

which completes the proof. ◊

Chapter 4

Permit price dynamics (Ordinary scheme)

4.1 Jumpy behaviour

4.1a Introduction

The permit price series of the EU ETS (European Union Emissions Trading Scheme) is characterized, especially, during the first phase by a jumpy behaviour and a massive price slump of about -50% within only two weeks (28 April 2006 - 15 May 2006). Therefore, the following classes of processes have been used to describe the price dynamics: jump-diffusion models (Wagner (2007), Daskalakis et al. (2009)), GARCH-models (Benz and Trück (2008) and Wagner (2007)), regime-switching models (Wagner (2007), Benz and Trück (2008)), Mix-Normal GARCH-models (Paolella and Taschini (2008)) and two-factor models (Cetin and Verschuere (2008)). Other authors support the argument that the permit price responds to macroeconomic fundamentals and try to explain the price evolution of emission permits in terms of electricity, gas, oil and coal prices and weather effects (cf. Hintermann (2009) and Mansanet-Bataller et al. (2007)).
However, none of the papers is able to explain why permit prices are extremely jumpy. Therefore, this question is discussed in the following subsections from a theoretical point of view. The permit price models in Section 3.2c and 3.2d capture the main characteristics of an ordinary scheme such as Phase I of the EU ETS . The models of Chesney and Taschini (2008) and Grüll and Kiesel (2009) are of special interest as analytical permit price formulae are available for these models.

Therefore, the theoretical analysis of the jumpy permit price behaviour is performed in the framework of Chesney and Taschini (2008) and Grüll and Kiesel (2009). Section 3.2c and 3.2d presented three different permit price formulae in the models of Chesney and Taschini (2008) and Grüll and Kiesel (2009):

- Linear approximation (cf. Lemma 3.2.16 in Section 3.2c)
- Log-normal moment matching (cf. Lemma 3.2.22 in Section 3.2d)
- Reciprocal gamma moment matching (cf. Lemma 3.2.23 in Section 3.2d)

In Section 4.1b it is shown how the theoretical permit price is related to a random variable that is very easy to interpret (cf. Definition 4.1.1). These results are illustrated in Section 4.1c. Section 4.1d provides a theoretical explanation for the jumpy price behaviour. The derivation of this result makes use of the theorems in Section 4.1b.

4.1b Permit price properties

Definition 4.1.1 (Time needed to exhaust the remaining permits)
Let Q_t be the emission rate at time t, $q_{[0,t]}$ the cumulative emissions until t and N the number of allowances handed out by the regulator.
Then the time needed to exhaust the remaining permits at the current emission rate is defined by

$$x_t = \frac{N - q_{[0,t]}}{Q_t}. \tag{4.1}$$

Lemma 4.1.2 (Permit price properties - Linear approximation)
The permit price for the linear approximation is given by

$$S_t^{Lin} = \begin{cases} Pe^{-r\tau} & \text{if } q_{[0,t]} \geq N \\ Pe^{-r\tau} \cdot \Phi\left(\frac{-\ln\left(\frac{1}{\tau}\left[\frac{N-q_{[0,t]}}{Q_t}\right]\right)+\left(\mu-\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right) & \text{if } q_{[0,t]} < N \end{cases} \tag{4.2}$$

where $\tau = T - t$ is the time to compliance.
(a) Let $t \in [0,T)$, $c > 0$ and $q_{[0,t]} < N$. Then
$S_t^{Lin} \in (Pe^{-r\tau} \cdot \Phi(-c), Pe^{-r\tau} \cdot \Phi(c))$ iff $x_t \in (a_t, b_t)$ where

$$a_t := \tau \exp\left\{-c\sigma\sqrt{\tau} + \left(\mu - \frac{\sigma^2}{2}\right)\tau\right\},$$

$$b_t := \tau \exp\left\{c\sigma\sqrt{\tau} + \left(\mu - \frac{\sigma^2}{2}\right)\tau\right\}.$$

(b) Let $t \in [0,T)$ *and* $q_{[0,t]} < N$. *Then*

$$\frac{dS_t^{Lin}}{dx_t}(x_t) := -\frac{Pe^{-r\tau}}{\sigma\sqrt{\tau}} \cdot \frac{1}{x_t}\phi\left(\frac{-\ln\left(\frac{1}{\tau}x_t\right) + \left(\mu - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right) < 0.$$

(c) Let $t \in [0,T)$ *and* $q_{[0,t]} < N$.
Then a change of h *per cent in* $x_t := \frac{N-q_{[0,t]}}{Q_t}$ *implies approximately a permit price change of* p *per cent*

$$p = -\frac{\phi\left(\frac{-\ln\left(\frac{1}{\tau}x_t\right)+\left(\mu-\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right)}{\Phi\left(\frac{-\ln\left(\frac{1}{\tau}x_t\right)+\left(\mu-\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right)} \cdot \frac{h}{\sigma\sqrt{\tau}}.$$

Proof :
(a) By Lemma 3.2.16

$$S_t^{Lin} < Pe^{-r\tau} \cdot \Phi(c) \Leftrightarrow Pe^{-r\tau} \cdot \Phi\left(\frac{-\ln\left(\frac{1}{\tau}\left[\frac{N-q_{[0,t]}}{Q_t}\right]\right) + \left(\mu - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right) < Pe^{-r\tau} \cdot \Phi(c)$$

$$\Leftrightarrow \frac{-\ln\left(\frac{1}{\tau}\left[\frac{N-q_{[0,t]}}{Q_t}\right]\right) + \left(\mu - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} < c$$

$$\Leftrightarrow \frac{\ln\left(\frac{1}{\tau}\left[\frac{N-q_{[0,t]}}{Q_t}\right]\right) - \left(\mu - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} > -c$$

$$\Leftrightarrow x_t > \tau \exp\left\{-c\sigma\sqrt{\tau} + \left(\mu - \frac{\sigma^2}{2}\right)\tau\right\}.$$

Analogous to the above proof we have

$$S_t^{Lin} > Pe^{-r\tau} \cdot \Phi(-c) \Leftrightarrow x_t < \tau \exp\left\{c\sigma\sqrt{\tau} + \left(\mu - \frac{\sigma^2}{2}\right)\tau\right\}.$$

(b) Follows from deriving Equation (4.2).
(c) The following expression

$$\frac{dS_t^{Lin}}{dx_t}(x_t) \approx \frac{S_t^{Lin}((1+h)x_t) - S_t^{Lin}(x_t)}{hx_t}$$

is equivalent to

$$\frac{S_t^{Lin}((1+h)x_t) - S_t^{Lin}(x_t)}{S_t^{Lin}(x_t)} \approx hx_t\frac{\frac{dS_t^{Lin}}{dx_t}(x_t)}{S_t^{Lin}(x_t)}.$$

Lemma 3.2.16 and part (a) complete the proof. $\diamondsuit$

Lemma 4.1.3 (Permit price properties - Log-normal moment matching)
The permit price in the log-normal moment matching approach is given by

$$S_t^{Log} = \begin{cases} Pe^{-r\tau} & \text{if } q_{[0,t]} \geq N \\ Pe^{-r\tau} \cdot \Phi\left(\frac{-\ln\left(\frac{N-q_{[0,t]}}{Q_t}\right)+2\ln(\alpha_\tau)-\frac{1}{2}\ln(2\beta_\tau)}{\sqrt{\ln(2\beta_\tau)-2\ln(\alpha_\tau)}}\right) & \text{if } q_{[0,t]} < N \end{cases} \tag{4.3}$$

where $\tau = T - t$ *is the time to compliance and* α_τ, β_τ *are given in Lemma 3.2.19.*
(a) Let $t \in [0, T)$, $c > 0$ *and* $q_{[0,t]} < N$. *Then*
$S_t^{Log} \in (Pe^{-r\tau} \cdot \Phi(-c), Pe^{-r\tau} \cdot \Phi(c))$ *iff* $x_t \in (a_t, b_t)$ *where*

$$\begin{aligned} a_t &:= \exp\left\{-c\sqrt{\ln(2\beta_\tau) - 2\ln(\alpha_\tau)} + 2\ln(\alpha_\tau) - \frac{1}{2}\ln(2\beta_\tau)\right\}, \\ b_t &:= \exp\left\{c\sqrt{\ln(2\beta_\tau) - 2\ln(\alpha_\tau)} + 2\ln(\alpha_\tau) - \frac{1}{2}\ln(2\beta_\tau)\right\}. \end{aligned}$$

(b) Let $t \in [0, T)$ *and* $q_{[0,t]} < N$. *Then*

$$\frac{dS_t^{Log}}{dx_t}(x_t) := -\frac{Pe^{-r\tau}}{\sqrt{\ln(2\beta_\tau) - 2\ln(\alpha_\tau)}} \cdot \frac{1}{x_t}\phi\left(\frac{-\ln(x_t) + 2\ln(\alpha_\tau) - \frac{1}{2}\ln(2\beta_\tau)}{\sqrt{\ln(2\beta_\tau) - 2\ln(\alpha_\tau)}}\right) < 0.$$

(c) Let $t \in [0, T)$ *and* $q_{[0,t]} < N$.
Then a change of h *per cent in* $x_t := \frac{N-q_{[0,t]}}{Q_t}$ *implies approximately a permit price change of* p *per cent*

$$p = -\frac{\phi\left(\frac{-\ln(x_t)+2\ln(\alpha_\tau)-\frac{1}{2}\ln(2\beta_\tau)}{\sqrt{\ln(2\beta_\tau)-2\ln(\alpha_\tau)}}\right)}{\Phi\left(\frac{-\ln(x_t)+2\ln(\alpha_\tau)-\frac{1}{2}\ln(2\beta_\tau)}{\sqrt{\ln(2\beta_\tau)-2\ln(\alpha_\tau)}}\right)} \cdot \frac{h}{\sqrt{\ln(2\beta_\tau) - 2\ln(\alpha_\tau)}}.$$

Proof :
(a) By Lemma 3.2.22

$$\begin{aligned} S_t^{Log} < Pe^{-r\tau} \cdot \Phi(c) &\Leftrightarrow Pe^{-r\tau} \cdot \Phi\left(\frac{-\ln\left(\frac{N-q_{[0,t]}}{Q_t}\right) + 2\ln(\alpha_\tau) - \frac{1}{2}\ln(2\beta_\tau)}{\sqrt{\ln(2\beta_\tau) - 2\ln(\alpha_\tau)}}\right) < Pe^{-r\tau} \cdot \Phi(c) \\ &\Leftrightarrow \frac{-\ln\left(\frac{N-q_{[0,t]}}{Q_t}\right) + 2\ln(\alpha_\tau) - \frac{1}{2}\ln(2\beta_\tau)}{\sqrt{\ln(2\beta_\tau) - 2\ln(\alpha_\tau)}} < c \\ &\Leftrightarrow \frac{\ln\left(\frac{N-q_{[0,t]}}{Q_t}\right) - 2\ln(\alpha_\tau) + \frac{1}{2}\ln(2\beta_\tau)}{\sqrt{\ln(2\beta_\tau) - 2\ln(\alpha_\tau)}} > -c \\ &\Leftrightarrow x_t > \exp\left\{-c\sqrt{\ln(2\beta_\tau) - 2\ln(\alpha_\tau)} + 2\ln(\alpha_\tau) - \frac{1}{2}\ln(2\beta_\tau)\right\}. \end{aligned}$$

Analogous to the above proof we have

$$S_t^{Log} > Pe^{-r\tau} \cdot \Phi(-c) \Leftrightarrow \; x_t < \exp\left\{ c\sqrt{\ln(2\beta_\tau) - 2\ln(\alpha_\tau)} + 2\ln(\alpha_\tau) - \frac{1}{2}\ln(2\beta_\tau) \right\}.$$

(b) Follows from deriving Equation (4.3).

(c) Analogous to the proof of Lemma 3.2.16 (c). ♢

Lemma 4.1.4 (Permit price properties - Reciprocal gamma moment matching)

The permit price for the reciprocal gamma moment matching approach is given by

$$S_t^{IG} = \begin{cases} Pe^{-r\tau} & \text{if } q_{[0,t]} \geq N \\ Pe^{-r\tau} \cdot G\left(\frac{Q_t}{N - q_{[0,t]}} | \frac{4\beta_\tau - \alpha_\tau^2}{2\beta_\tau - \alpha_\tau^2}, \frac{2\beta_\tau - \alpha_\tau^2}{2\alpha_\tau \beta_\tau} \right) & \text{if } q_{[0,t]} < N \end{cases} \tag{4.4}$$

where $\tau = T - t$ is the time to compliance and α_τ, β_τ are given in Lemma 3.2.19. $G(x|\alpha, \beta)$ denotes the c.d.f. of a gamma distributed random variable with location parameter α and scale parameter β.

(a) Let $t \in [0, T)$ and $q_{[0,t]} < N$. Then

$$\frac{dS_t^{IG}}{dx_t}(x_t) := \; -Pe^{-r\tau} g_R\left(x_t | \frac{4\beta_\tau - \alpha_\tau^2}{2\beta_\tau - \alpha_\tau^2}, \frac{2\beta_\tau - \alpha_\tau^2}{2\alpha_\tau \beta_\tau} \right) < 0.$$

(b) Let $t \in [0, T)$ and $q_{[0,t]} < N$.

Then a change of h per cent in $x_t := \frac{N - q_{[0,t]}}{Q_t}$ implies approximately a permit price change of p per cent

$$p = -\frac{g_R\left(x_t | \frac{2\beta_\tau}{2\beta_\tau - \alpha_\tau^2}, \frac{2\beta_\tau - \alpha_\tau^2}{2\alpha_\tau \beta_\tau} \right)}{G\left(\frac{1}{x_t} | \frac{4\beta_\tau - \alpha_\tau^2}{2\beta_\tau - \alpha_\tau^2}, \frac{2\beta_\tau - \alpha_\tau^2}{2\alpha_\tau \beta_\tau} \right)} \cdot \alpha_\tau h.$$

Proof :

(a) For $N > q_{[0,t]}$ differentiating $\hat{S}_t(x_t)$ yields

$$\begin{aligned} \frac{dS_t^{IG}}{dx_t}(x_t) = & \; - Pe^{-r\tau} \frac{1}{x_t^2} \cdot g\left(\frac{1}{x_t} | \frac{4\beta_\tau - \alpha_\tau^2}{2\beta_\tau - \alpha_\tau^2}, \frac{2\beta_\tau - \alpha_\tau^2}{2\alpha_\tau \beta_\tau} \right) \\ \overset{(3.77)}{=} & \; - Pe^{-r\tau} \cdot g_R\left(x_t | \frac{4\beta_\tau - \alpha_\tau^2}{2\beta_\tau - \alpha_\tau^2}, \frac{2\beta_\tau - \alpha_\tau^2}{2\alpha_\tau \beta_\tau} \right) \end{aligned}$$

(b) Analogous to the proof of Lemma 3.2.16 (c). Hereby we use Equation (3.73) and (3.77) and the fact that $\frac{4\beta_\tau - \alpha_\tau^2}{2\beta_\tau - \alpha_\tau^2} > 1$. ♢

4.1c Numerical illustrations of equilibrium permit price

The theoretical time-t permit price is a function of μ and σ (parameters of the emission rate) and of x_t, the time needed to exhaust the remaining permits. In case that we know x_t (equivalent to knowing $q_{[0,t]}$) the time-t permit price is deterministic. Otherwise the permit price is a function of the random variable x_t. The following graphs show trajectories of the permit price for the different approximation approaches. In order to make the prices for the different approximation approaches comparable we proceed as follows: we create a sample path for the emission rate which yields a sample path for the cumulative emissions and x_t. Then, we compute the permit prices for the different approximation approaches using the same emissions trajectory. Furthermore, we plot the permit prices of the expected cumulative emissions $\mathbb{E}\left(q_{[0,t]}\right)$. Lower and upper bounds are obtained by considering the permit prices of $\mathbb{E}\left(q_{[0,t]}\right) \pm c\sqrt{Var\left(q_{[0,t]}\right)}$ which is the confidence band for cumulative emissions. Figure 4.1 shows the trajectory of x_t that is used to compute the price trajectories in Figure 4.2. It is important to note that the variance of the sample trajectory in Figure 4.1 is very small (almost behaving like an affine function) but even minor changes in x_t can lead to relatively large price jumps (cf. Figure 4.2). This hints at permit prices that are inherently prone to jumps (resulting from changes in the expected time needed to exhaust the remaining permits). Figure 4.2 and 4.3 illustrate that there is a difference between the price of the linear approximation approach and the prices of the moment matching approaches. The prices of the log-normal and the reciprocal gamma moment matching approach are very close to each other. Therefore, we focus on the log-normal moment matching approach and the linear approximation approach in Section 4.1d.

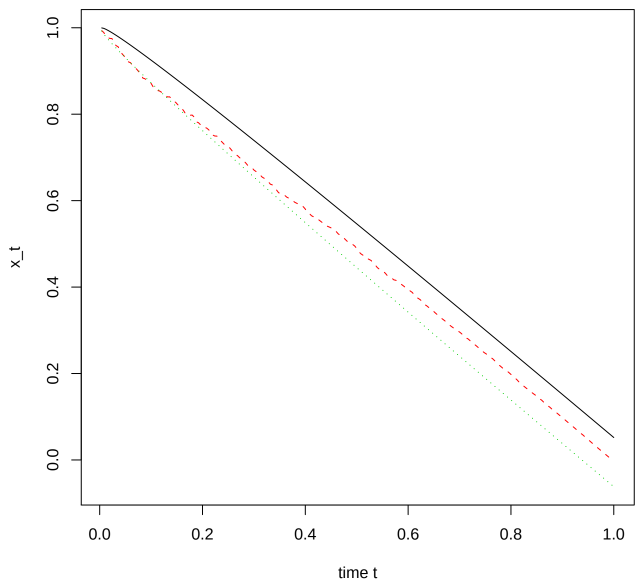

Figure 4.1: Trajectory of $x_t = \frac{N - q_{[0,t]}}{Q_t}$ with confidence interval for x_t (2 StDev). $t \in [0,1]$, $N = Q_0 = 100$, $\mu = 0.02$ and $\sigma = 0.05$.

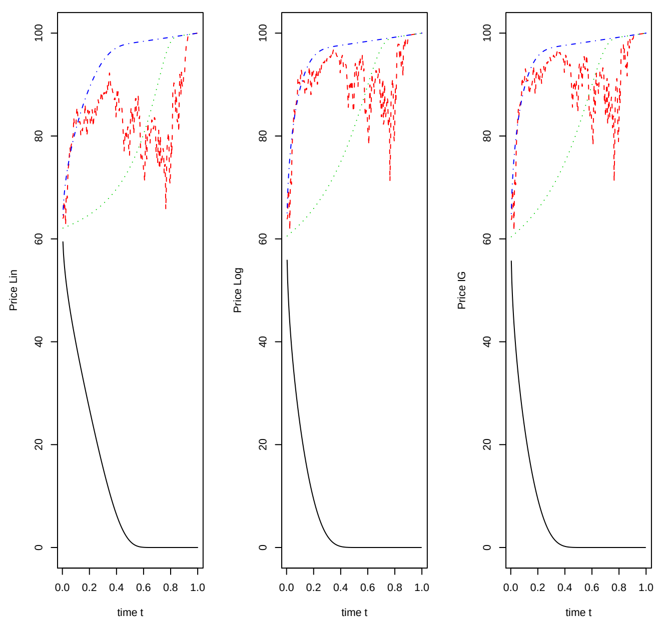

Figure 4.2: Trajectory of $S_t^{Lin}(x_t)$ (left), $S_t^{Log}(x_t)$ (middle) and $S_t^{IG}(x_t)$ (right) with price path for expected cumulative emissions and its lower and upper confidence band. $t \in [0,1]$, $N = Q_0 = 100$, $\mu = 0.02$ and $\sigma = 0.05$.

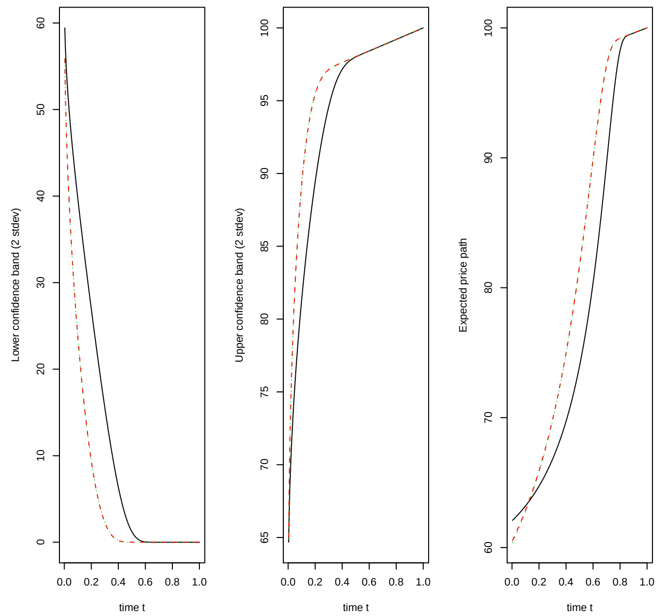

Figure 4.3: Price path for lower (Left) and upper (Middle) confidence band for cumulative emissions and for expected cumulative emissions (Right). Moment matching approaches (straight lines); Linear approximation (dashed line). $t \in [0,1]$, $N = Q_0 = 100$, $\mu = 0.02$ and $\sigma = 0.05$.

4.1d Implied parameters

The preceding subsection illustrated the behaviour of the permit prices using simulated data. With the aim to analyze the observed permit prices within a stochastic equilibrium model we look at the following two useful implied parameters.

1. Implied time needed to exhaust the remaining permits x_t^{impl}
2. Implied adjustment of x_t due to the announcement of verified emissions and the observed resulting permit price change

The implied time needed to exhaust the remaining permits x_t^{impl} is obtained by equating the observed permit price S_t and the theoretical permit price $S_t^{approx}\left(x_t^{impl}|\mu,\sigma\right)$ and then solving for x_t^{impl} . Hereby we fix the parameters μ and σ of the emission rate. Performing a scenario analysis by varying the parameters μ and σ allows us to relate the permit price level to x_t, the time needed to exhaust the remaining permits. An advantage of x_t is that it is easy to interpret and that it can be used in the context of crosschecks. Moreover, it is much handier than the interpretation of the permit price as the discounted penalty times the probability of permit shortage at compliance time. Based on the implied time needed to exhaust the remaining permits we define the following random variable.

Definition 4.1.5 (Over-/Underallocation)
Let x_t be the time needed to exhaust the remaining permits as defined in Definition 4.1.1 and let $T-t$ be the time to compliance.
Then over-/underallocation is defined by

$$x_t - (T-t). \tag{4.5}$$

Remark:
Over-/underallocation is measured in years. Positive (negative) values correspond to overallocation (underallocation).

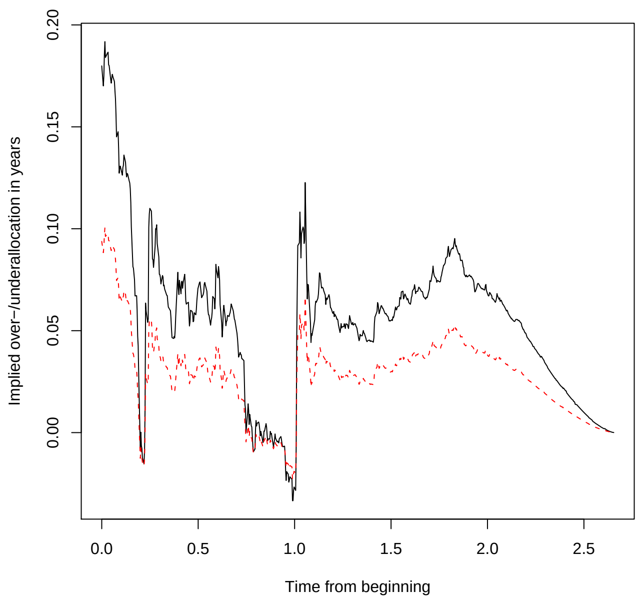

Figure 4.4: Implied over-/underallocation for first phase of EU ETS for fixed $\mu = 0.02$ and $\sigma = 0.05$. Linear approximation approach (straight line), log-normal moment matching (dashed line).

Figure 4.4 shows the implied over-/underallocation for the first phase of the EU ETS (May 2005 - December 2007). In the first time period (May 2005 - April 2006) the permit prices implied that there was an overallocation of permits. However, the level of implied overallocation declined and reached slight implied underallocation in March/April 2006. Rumours of overallocation and the final official publication of the verified emission data for 2005 on 15 May 2006 (overallocation of 2.5 per cent in 2005) lead to a jump from slight implied underallocation to implied overallocation. In the time period from 15 May 2006 until compliance time at the end of 2007 the market sentiment did not change and remained in the status of implied overallocation. Note that the two approximation approaches lead to different values of x_t^{impl} with the log-normal moment-matching approach being more sensitive to changes in x_t. Nevertheless, both approximation approaches lead to the same interpretation of overallocation and underallocation during the first phase of the EU ETS.

As the price slump of about -50% between 28 April 2006 and 15 May 2006 is one of the main characteristics of the permit price time series of the first EU ETS phase we extensively analyze the relationship of the permit price and the implied adjustment of the time needed to exhaust the remaining permits x_t (cf. Lemma 4.1.6 and Lemma 4.1.7). Using the information of the announcement of the verified emissions and the permit prices both before and after the publication of the verified emission data, Lemma 4.1.6 gives us the implied percentage change of x_t for fixed volatility σ of the emission rate.
Figure 4.5 shows that for reasonable values of σ the extreme permit price drop is caused by a relatively small adjustment of x_t. The log-normal moment matching approach is even more sensitive towards a change of x_t than the linear approximation approach as already observed in Figure 4.4.

Lemma 4.1.6 (Determining implied parameters)
Let $\bar{x}_t = \left(\frac{N}{q_{[0,t]}} - 1\right) t$ *be the estimator of* x_t *on the day of the announcement of the cumulative emissions.*
Assume that we know the number of permits N *, the cumulative emissions at time t,* $q_{[0,t]}$ *, and the futures permit prices both at time* t *and* $t - \Delta$ *denoted by* $F(t,T)$ *and* $F(t-\Delta,T)$ *, respectively.*
Moreover, assume that σ *is given. Then*

1. *In the framework of the linear approximation approach*
   (a) the implied parameter $\mu(\sigma)$ *is approximately*

$$\mu(\sigma) = \frac{1}{2}\sigma^2 + \frac{\Phi^{-1}\left(\frac{F(t,T)}{P}\right)}{\sqrt{\tau}} \cdot \sigma + \frac{\ln\left(\frac{\bar{x}_t}{\tau}\right)}{\tau}. \tag{4.6}$$

   (b) the implied time needed to exhaust the remaining permits before the announcement of the remaining permits is approximately h *per cent larger than* $\bar{x}_t$ *:*

$$h = -\frac{[F(t-\Delta,T) - F(t,T)]}{P\phi\left(\Phi^{-1}\left(\frac{F(t,T)}{P}\right)\right)} \cdot (\sigma\sqrt{\tau}). \tag{4.7}$$

2. *In the framework of the log-normal moment matching approach*
   (a) the implied parameter $\mu(\sigma)$ *is approximately the solution of*

$$\Phi^{-1}\left(\frac{F(t,T)}{P}\right) = \frac{-\ln(\bar{x}_t) + 2\ln(\alpha_\tau) - \frac{1}{2}\ln(2\beta_\tau)}{\sqrt{\ln(2\beta_\tau) - 2\ln(\alpha_\tau)}}. \tag{4.8}$$

   (b) the implied time needed to exhaust the remaining permits before the announcement of the remaining permits is approximately h *per cent larger than* $\bar{x}_t$ *:*

$$h = -\frac{[F(t-\Delta,T) - F(t,T)]}{P\phi\left(\Phi^{-1}\left(\frac{F(t,T)}{P}\right)\right)} \cdot \sqrt{\ln(2\beta_\tau) - 2\ln(\alpha_\tau)}, \tag{4.9}$$

   where $\alpha_\tau(\mu(\sigma),\sigma)$ *and* $\beta_\tau(\mu(\sigma),\sigma)$ *solve Equation (4.8).*

Proof :
For small μ and σ the emission rate Q_t can be approximated by Q_0 . Thus

$$q_{[0,t]} = \int_0^t Q_s ds \approx \int_0^t Q_0 ds = t \cdot Q_0.$$

Hence

$$x_t = \frac{N - q_{[0,t]}}{Q_t} \approx \frac{N - q_{[0,t]}}{Q_0} \approx \left(\frac{N}{q_{[0,t]}} - 1\right) \cdot t := \bar{x}_t.$$

As the proof for both approximation approaches is the same we only present the proof of Equation (4.6) and (4.7).

(a): Assuming that the observed permit price S_t can be described by the equilibrium price formula of the linear approximation approach in Lemma 3.2.16 and solving $S_t = S_t^{Lin}(\bar{x}_t|\mu,\sigma)$ for μ yields Equation (4.6).

(b): Assuming that $S_{t-\Delta} = S_{t-\Delta}^{Lin}(\bar{x}_t(1+h)|\mu,\sigma) \approx S_t^{Lin}(\bar{x}_t(1+h)|\mu,\sigma)$ and using

$$S_t^{Lin}(\bar{x}_t(1+h)|\mu,\sigma) \approx S_t^{Lin}(\bar{x}_t|\mu,\sigma) + h\bar{x}_t \frac{dS_t^{Lin}}{dx_t}(\bar{x}_t|\mu,\sigma)$$

yields

$$S_{t-\Delta} - S_t \approx h\bar{x}_t \frac{dS_t^{Lin}}{dx_t}(\bar{x}_t|\mu,\sigma).$$

As the implied parameter $\mu(\sigma)$ satisfies the following equation

$$\frac{-\ln\left(\frac{\bar{x}_t}{\tau}\right) + \left(\mu(\sigma) - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} = \Phi^{-1}\left(\frac{F(t,T)}{P}\right)$$

Lemma 3.2.16 (c) completes the proof. $\diamondsuit$

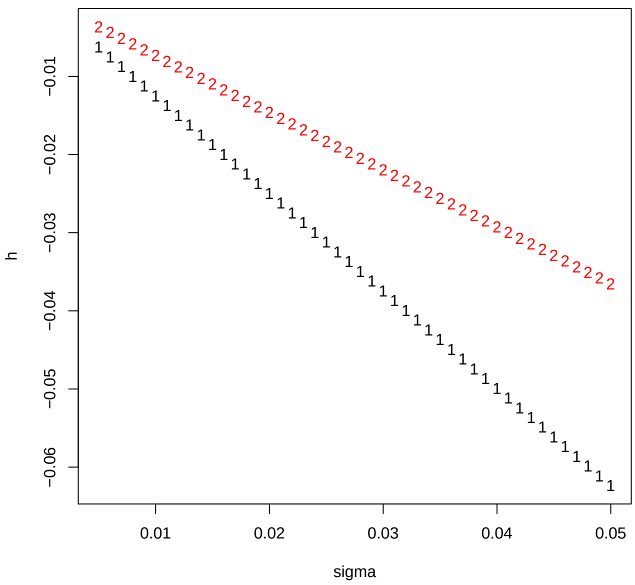

Figure 4.5: Before the revelation of the cumulative emissions the implied time to exhaust the remaining permits was $-h(\sigma)$ per cent smaller. Linear approximation approach ("1"), log-normal moment matching ("2").

Lemma 4.1.7 (Determining implied parameter - emission rate is martingale)

Assume that the emission rate is a martingale, i.e. $\mu = 0$ *.*
Let $\bar{x}_t = \left(\frac{N}{q_{[0,t]}} - 1\right) t$ *be the estimator of* x_t *on the day of the announcement of the cumulative emissions.*
Assume that we know the number of permits N *, the cumulative emissions at time t,* $q_{[0,t]}$ *, and the futures permit prices both at time t and at time* $t - \Delta$ *denoted by* $F(t,T)$ *and* $F(t-\Delta,T)$ *, respectively. Then the implied time needed to exhaust the remaining permits before the announcement is approximately* h *per cent larger than* $\bar{x}_t$ *:*

$$h = -\frac{[F(t-\Delta,T) - F(t,T)]}{P\phi\left(\Phi^{-1}\left(\frac{F(t,T)}{P}\right)\right)}\gamma.$$

where γ *depends on the approximation approach and is given by the positive solution of the following equation (in case it exists)*

1. *Linear approximation approach*

$$\gamma = 2\left[-\Phi^{-1}\left(\frac{F(t,T)}{P}\right) \pm \sqrt{\left(\Phi^{-1}\left(\frac{F(t,T)}{P}\right)\right)^2 - 2\ln\left(\frac{\bar{x}_t}{\tau}\right)}\right]$$

2. *Log-normal moment matching*

$$\gamma = 2\left[-\Phi^{-1}\left(\frac{F(t,T)}{P}\right) \pm \sqrt{\left(\Phi^{-1}\left(\frac{F(t,T)}{P}\right)\right)^2 - 2\ln(\bar{x}_t)}\right]$$

Proof :
Follows from Lemma 6 (b) and the equilibrium price formula of Lemma 3.2.16 and of Lemma 3.2.22, respectively. ♢

Remark:
Lemma 4.1.7 cannot be used to explain the price drop in 2006 as there exists no solution of γ for the permit price series of the first phase of the EU ETS.

4.1e Conclusion

The numerical illustrations in Section 4.1c and 4.1d show that the choice of the approximation approach for the cumulative emissions influences the resulting theoretical permit price level significantly. The prices of the log-normal and the reciprocal gamma moment matching approach are very close to each other but there is a noticeable difference between the prices of the moment matching approaches and the linear approximation approach as used by Chesney and Taschini (2008). Therefore, it does not suffice to state the process that models the emission rate. It is essential to specify both the process for the emission rate and the applied approximation approach.

Nevertheless, all the three different approximation approaches have in common that the permit price is inherently prone to jumps as shown in Section 4.1c - 4.1d. The extreme price slump in 2006 can be explained in our equilibrium model by a relatively small change in the market's expectation for how long the remaining permits will suffice. It is shown that the moment-matching approach is even more sensitive to changes in the market's sentiment of over-/underallocation than the linear approximation approach. Given a permit price time series we can compute the market's implied expectation of over- or underallocation using our equilibrium price formulae. Therefore, the price drop in 2006 of about -50% can be explained as follows: With the market being in slight implied underallocation in April 2006 (cf. Section 4.1d) rumours of a probable overallocation and the final publication of the verified emission data for 2005 on 15 May 2006 by the European Commission (overallocation of 2.5%) drove prices lower. Price jumps will occur as long as the market's ability to estimate the cumulative emission level is limited. A nearly exact estimation must have been impossible until the publication of the first verified emission data in May 2006. Even the regulator could not aggregate the data of all the countries for the first emissions report - emission data for the Czech Republic, France, the Slovak Republic and Spain was partly missing (cf. European Union (2006)). However, price jumps of the magnitude of 2006 are unlikely to occur again as the measurement of the emission data has been significantly improved.

The scheme design is not the only source of price volatility. Emissions trading schemes are surrounded by regulatory risks. Changes in the regulation or even expected or feared changes have a significant influence on prices. As pointed out in Section 2.2b and 2.2g in detail, for instance the following two regulatory risks have been responsible for permit price slumps in the EU ETS: (i) changes in the cap even during the compliance period and (ii) influence of the reduction commitments of other countries on the reduction target of the European Union.

4.2 Convergence to zero

The concept of probability of shortage resulting from stochastic equilibrium models explains that prices in an emissions trading system without banking will always converge to zero or to the penalty at the end of the compliance period.
At the end of the compliance period there is no uncertainty about the cumulative emissions in the compliance period. Therefore, the probability of shortage only takes the values zero or one at the end of the compliance period. Multiplying this probability with the penalty yields the permit price in a stochastic equilibrium model.

In mathematical terms, this result follows from the permit price formula of Carmona et al. (2009b) as given in Equation (3.51):

$$\begin{aligned} S_T &= Pe^{-r(T-T)} \cdot \mathbb{P}\left(q_{[0,T]} > N|\mathcal{F}_T\right) \\ &= P \cdot \mathbb{P}\left(q_{[0,T]} > N|\mathcal{F}_T\right) \\ &= P \cdot \mathbf{1}_{\left\{q_{[0,T]}>N\right\}} \\ &= \begin{cases} P & \text{if } q_{[0,T]} > N \\ 0 & \text{if } q_{[0,T]} \leq N \end{cases} \end{aligned}$$

Analogous results hold in the framework of Chesney and Taschini (2008) and Grüll and Kiesel (2009). In the model of Seifert et al. (2008) this property follows from Theorem 3.2.8.

4.3 Discussion of estimation methods

4.3a Estimation methods for stochastic equilibrium models

This subsection investigates estimation methods for stochastic equilibrium models. The models of Chesney and Taschini (2008) and Grüll and Kiesel (2009) are of special interest because of the availability of analytical permit price formulae. Deriving an SDE for the permit price in the models of Chesney and Taschini (2008) and Grüll and Kiesel (2009) yields a very complicated expression that cannot be used for model calibration in practice. Therefore, we investigate two different approaches for the approximation of the random variable $\frac{N-q_{[0,t]}}{(T-t)\cdot Q_t}$ (cf. Lemma 4.3.2, 4.3.3 and 4.3.4).

Lemma 4.3.1 (Normalized permit price in the model of Chesney and Taschini)

The time-t permit price divided by the penalty is given by

$$a_t = \Phi\left(\frac{-\ln\left(\frac{N-q_{[0,t]}}{(T-t)\cdot Q_t}\right)+\left(\mu-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right).$$

In particular, we have

$$a_0 = \Phi\left(\frac{-\ln\left(\frac{N}{T\cdot Q_0}\right)+\left(\mu-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right).$$

Proof :
Follows directly from $a_t = \frac{F(t,T)}{P}$ and Lemma 3.2.16. ♢

Approximation 1:
A linear approximation in the nominator of $\frac{N-q_{[0,t]}}{(T-t)\cdot Q_t}$ yields $\ln\left(\frac{N-q_{[0,t]}}{(T-t)\cdot Q_t}\right) = \ln\left(\frac{N-t\cdot Q_t}{(T-t)\cdot Q_t}\right)$.
Now, approximating Q_t in the nominator by its expected value $\mathbb{E}\left[Q_t\right] = Q_0 e^{\mu t}$ yields

$$\ln\left(\frac{N-q_{[0,t]}}{(T-t)\cdot Q_t}\right) \approx \ln\left(\frac{N-t\cdot\mathbb{E}\left[Q_t\right]}{(T-t)\cdot Q_t}\right).$$

Theorem 4.3.2 (SDE for the model of Chesney&Taschini - Approximation 1)

Using the approximation $\ln\left(\frac{N-q_{[0,t]}}{(T-t)\cdot Q_t}\right) \approx \ln\left(\frac{N-t\cdot\mathbb{E}[Q_t]}{(T-t)\cdot Q_t}\right)$, *the dynamics of the permit price in the model of Chesney and Taschini (2008) are given by*

$$da_t = -\frac{\Phi'\left(\Phi^{-1}(a_t)\right)}{\sqrt{T-t}}\left[\left(\frac{(1+\mu t)Q_0 e^{\mu t}}{\sigma(N-t\cdot Q_0 e^{\mu t})}-\frac{1}{T-t}\right)dt + dW_t\right].$$

Proof :
Using the approximation $q_{[0,t]} = t\cdot Q_t$, we get

$$\begin{aligned}
a_t &= \Phi\left(\frac{-\ln\left(\frac{N-t\cdot Q_t}{(T-t)\cdot Q_t}\right)+\left(\mu-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\\
&= \Phi\left(\frac{-\ln\left(N-t\cdot Q_t\right)+\ln(T-t)+\ln(Q_t)+\left(\mu-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\\
&= \Phi\left(\frac{-\ln\left(N-t\cdot Q_t\right)+\ln(T-t)+\ln(Q_0)+\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma W_t+\left(\mu-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \Phi\left(\frac{-\ln\left(N - t\cdot Q_t\right) + \ln(T-t) + \ln(Q_0) + \sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T-t}}\right) \\
&= \Phi\left(\frac{\Phi^{-1}(a_0)\sigma\sqrt{T} + \ln\left(\frac{N}{T}\right) - \ln\left(N - t\cdot Q_t\right) + \ln(T-t) + \sigma W_t}{\sigma\sqrt{T-t}}\right).
\end{aligned}
$$

Now, using the approximation $Q_t \approx \mathbb{E}\left[Q_t\right] = Q_0 e^{\mu t}$ and plugging in $\ln\left(N - t\cdot Q_t\right)$, yields

$$
\begin{aligned}
a_t &= \Phi\left(\frac{\Phi^{-1}(a_0)\sigma\sqrt{T} + \ln\left(\frac{N}{T}\right) - \ln\left(N - t\cdot Q_0 e^{\mu t}\right) + \ln(T-t) + \sigma W_t}{\sigma\sqrt{T-t}}\right) \\
&:= \Phi\left(X_t\right) := \Phi\left(\frac{z(t)}{\sqrt{n(t)}}\right).
\end{aligned}
$$

The differential of the normalized permit price is

$$
\begin{aligned}
da_t &= d\Phi\left(X_t\right) = \Phi'(X_t)dX_t + \frac{1}{2}\Phi''(X_t)d[X]_t \\
&= \Phi'(X_t)dX_t - \frac{1}{2}X_t\Phi'(X_t)d[X]_t \\
&= \Phi'(X_t)\left[dX_t - \frac{1}{2}X_t d[X]_t\right],
\end{aligned}
$$

where

$$
\begin{aligned}
dn_t &= -\sigma^2 dt, \\
dz_t &= \left(\frac{(1+\mu t)Q_0 e^{\mu t}}{N - t\cdot Q_0 e^{\mu t}} - \frac{1}{T-t}\right)dt - \sigma dW_t, \\
dX_t &= \frac{1}{\sqrt{n_t}}dz_t - \frac{1}{2}\frac{X_t}{n_t}dn_t \\
&= \frac{1}{\sigma\sqrt{T-t}}\left(\frac{(1+\mu t)Q_0 e^{\mu t}}{N - t\cdot Q_0 e^{\mu t}} - \frac{1}{T-t}\right)dt - \frac{1}{\sqrt{T-t}}dW_t + \frac{1}{2}\frac{X_t}{T-t}dt, \\
d[X]_t &= \frac{1}{T-t}dt.
\end{aligned}
$$

Thus

$$
dX_t - \frac{1}{2}X_t d[X]_t = \left(\frac{1}{\sigma\sqrt{T-t}}\frac{(1+\mu t)Q_0 e^{\mu t}}{N - t\cdot Q_0 e^{\mu t}} - \frac{1}{T-t}\right)dt - \frac{1}{\sqrt{T-t}}dW_t.
$$

$\diamondsuit$

Approximation 2:

Bearing in mind that $T-t$ is an affine function and that the number of remaining permits is approximately an affine function in t, we can use the following approximation for small positive Δ

$$\frac{N-q_{[0,t+\Delta]}}{N-q_{[0,t]}}\frac{T-t}{T-(t+\Delta)}\approx 1.$$

We apply approximation 2 both to the model of Chesney and Taschini (2008) (cf. Theorem 4.3.3) and to the model of Grüll and Kiesel (2009) (cf. Theorem 4.3.4).

Theorem 4.3.3 (SDE for the model of Chesney&Taschini - Approximation 2)

Let $\frac{N-q_{[0,t+\Delta]}}{N-q_{[0,t]}}\frac{T-t}{T-(t+\Delta)}\approx 1$ *for small positive* Δ*. Then the following difference is approximately standard normally distributed in the model of Chesney and Taschini (2008)*

$$\frac{1}{\sqrt{\Delta}}\left(\Phi^{-1}(a_{t+\Delta})\sqrt{T-(t+\Delta)}-\Phi^{-1}(a_t)\sqrt{T-t}\right). \tag{4.10}$$

Proof :

By Lemma 4.3.1

$$\Phi^{-1}(a_t)\sqrt{T-t}=\frac{1}{\sigma}\cdot\left(-\ln\left(\frac{N-q_{[0,t]}}{(T-t)\cdot Q_t}\right)+\left(\mu-\frac{\sigma^2}{2}\right)(T-t)\right).$$

Thus,

$$\begin{aligned}
&\Phi^{-1}(a_t)\sqrt{T-t}-\Phi^{-1}(a_{t+\Delta})\sqrt{T-(t+\Delta)}\\
&\quad=\frac{1}{\sigma}\cdot\left(\ln\left(\frac{N-q_{[0,t+\Delta]}}{N-q_{[0,t]}}\cdot\frac{T-t}{T-(t+\Delta)}\right)-\ln\left(\frac{Q_{t+\Delta}}{Q_t}\right)+\left(\mu-\frac{\sigma^2}{2}\right)\Delta\right)\\
&\quad=\frac{1}{\sigma}\cdot\left(\ln\left(\frac{N-q_{[0,t+\Delta]}}{N-q_{[0,t]}}\cdot\frac{T-t}{T-(t+\Delta)}\right)-\sigma W_\Delta\right).
\end{aligned}$$

Assuming $\frac{N-q_{[0,t+\Delta]}}{N-q_{[0,t]}}\cdot\frac{T-t}{T-(t+\Delta)}\approx 1$ completes the proof. ♢

Theorem 4.3.4 (Discretized SDE for the model of Grüll and Kiesel)
Let $\frac{N-q_{[0,t+\Delta]}}{N-q_{[0,t]}}\frac{\frac{1}{\mu}(e^{\mu(T-t)}-1)}{\frac{1}{\mu}(e^{\mu(T-(t+\Delta))}-1)}\approx 1$ *for small positive* Δ *and let* $Z\sim N(0,1)$. *Then*

1. *The dynamics of the permit price in the model of Grüll and Kiesel (2009) are described by the following discretized SDE*

$$\begin{aligned} z_t &:= \Phi^{-1}(a_{t+\Delta})\sqrt{T-(t+\Delta)}-\Phi^{-1}(a_t)\sqrt{T-t} \\ &\sim N\left(\frac{\Delta}{\sqrt{b(\mu,\sigma^2)}}\left(\mu-\frac{\sigma^2}{2}+\frac{b(\mu,\sigma^2)}{2}\right),\frac{\sigma^2\Delta}{b(\mu,\sigma^2)}\right), \end{aligned}$$

where

$$b(\mu,\sigma^2)=\frac{\mu(\mu+\sigma^2)\left(e^{2\mu+\sigma^2}-e^{\mu}\right)}{\mu e^{2\mu+\sigma^2}+\mu+\sigma^2-(2\mu+\sigma^2)\,e^{\mu}}-2\frac{\mu e^{\mu}}{e^{\mu}-1}.$$

2. *Let* m *and* s^2 *be the sample mean and the sample variance of the data set* $\{z_t\}$. *Then the parameter estimate* $\hat{\sigma}^2$ *is given by the solution of*

$$b\left(\frac{m}{s\sqrt{\Delta}}\hat{\sigma}+\frac{1}{2}\left(1-\frac{\Delta}{s^2}\right)\hat{\sigma}^2,\hat{\sigma}^2\right)=\frac{\Delta}{s^2}\hat{\sigma}^2, \tag{4.11}$$

and the estimate $\hat{\mu}:=\hat{\mu}(\hat{\sigma}^2)$ *is given by*

$$\hat{\mu}=\frac{m}{s\sqrt{\Delta}}\hat{\sigma}+\frac{1}{2}\left(1-\frac{\Delta}{s^2}\right)\hat{\sigma}^2. \tag{4.12}$$

Proof :
(a) The permit price in the model of Grüll and Kiesel (2009) is given by

$$a_t=\Phi\left(\frac{-\ln\left(\frac{N-q_{[0,t]}}{Q_t}\right)+g(T-t)}{\sqrt{h(T-t)}}\right), \tag{4.13}$$

where

$$g(T-t)=\ln\left(\frac{\alpha_{T-t}^2}{\sqrt{2\beta_{T-t}}}\right),\quad\text{and}\quad h(T-t)=\ln\left(\frac{2\beta_{T-t}}{\alpha_{T-t}^2}\right). \tag{4.14}$$

Parameters α_{T-t} and β_{T-t} are given in Lemma 3.2.19.
The Taylor expansion around 1 yields

$$h(\tau)=h(1)+h'(1)(\tau-1)+\frac{1}{2}h''(\xi)(\xi-1)^2$$

for ξ between 1 and τ. It can be shown that the error term is sufficiently small for parameter combinations (μ,σ^2) that are in scope. Furthermore, it can be shown that $h(1)-h'(1)\approx 0$. Therefore, in the following we work with the approximation

$$h(T-t)\approx b(\mu,\sigma^2)(T-t),$$

where

$$b(\mu,\sigma^2) = h'(1) = \frac{\beta_1'}{\beta_1} - 2\frac{\alpha_1'}{\alpha_1}.$$

Thus

$$\begin{aligned}
h(T-t) \approx b(\mu,\sigma^2)(T-t) &\Leftrightarrow \frac{2\beta_{T-t}}{\alpha_{T-t}^2} \approx e^{b(\mu,\sigma^2)(T-t)} \\
&\Leftrightarrow \sqrt{2\beta_{T-t}} \approx \sqrt{e^{b(\mu,\sigma^2)(T-t)}}\alpha_{T-t} \\
&\Leftrightarrow \frac{\alpha_{T-t}^2}{\sqrt{2\beta_{T-t}}} \approx \frac{\alpha_{T-t}}{\sqrt{e^{b(\mu,\sigma^2)(T-t)}}} \\
&\Leftrightarrow g(T-t) \approx \ln(\alpha_{T-t}) - \frac{1}{2}b(\mu,\sigma^2)(T-t).
\end{aligned}$$

Inserting the approximation functions for $g(\cdot)$ and $h(\cdot)$ into Equation (4.13) yields

$$\Phi^{-1}(a_t) = \frac{1}{\sqrt{b(\mu,\sigma^2)(T-t)}}\left[-\ln\left(\frac{N-q_{[0,t]}}{Q_t}\right) + \ln(\alpha_{T-t}) - \frac{1}{2}b(\mu,\sigma^2)(T-t)\right],$$

which is equivalent to

$$\Phi^{-1}(a_t)\sqrt{T-t} = \frac{1}{\sqrt{b(\mu,\sigma^2)}}\left[-\ln\left(\frac{N-q_{[0,t]}}{Q_t}\right) + \ln(\alpha_{T-t}) - \frac{1}{2}b(\mu,\sigma^2)(T-t)\right].$$

For small positive Δ we have

$$\begin{aligned}
&\Phi^{-1}(a_{t+\Delta})\sqrt{T-(t+\Delta)} - \Phi^{-1}(a_t)\sqrt{T-t} \\
&= \frac{1}{\sqrt{b(\mu,\sigma^2)}}\left[\ln\left(\frac{N-q_{[0,t]}}{N-q_{[0,t+\Delta]}}\cdot\frac{\alpha_{T-(t+\Delta)}}{\alpha_{T-t}}\right) + \ln\left(\frac{Q_{t+\Delta}}{Q_t}\right) + \frac{\Delta}{2}b(\mu,\sigma^2)\right] \\
&= \frac{1}{\sqrt{b(\mu,\sigma^2)}}\left[\ln\left(\frac{N-q_{[0,t]}}{N-q_{[0,t+\Delta]}}\cdot\frac{\alpha_{T-(t+\Delta)}}{\alpha_{T-t}}\right) + \left(\mu - \frac{\sigma^2}{2} + \frac{b(\mu,\sigma^2)}{2}\right)\Delta + \sigma W_\Delta\right].
\end{aligned}$$

As both $N-q_{[0,t]}$ and α_{T-t} are approximately affine functions, we can use the following approximation

$$\frac{N-q_{[0,t]}}{N-q_{[0,t+\Delta]}}\cdot\frac{\alpha_{T-(t+\Delta)}}{\alpha_{T-t}} \approx 1$$

which completes the proof.

(b) We obtain the parameters $\hat{\mu}$ and $\hat{\sigma}^2$ by solving

$$m = \frac{\Delta}{\sqrt{b(\hat{\mu},\hat{\sigma}^2)}}\left(\hat{\mu} - \frac{\hat{\sigma}^2}{2} + \frac{b(\hat{\mu},\hat{\sigma}^2)}{2}\right), \tag{4.15}$$

$$s^2 = \frac{\hat{\sigma}^2\Delta}{b(\mu,\hat{\sigma}^2)}. \tag{4.16}$$

Solving Equation (4.16) for $b(\hat{\mu},\hat{\sigma}^2)$ and plugging the result into Equation (4.15) yields

$$\hat{\mu}(\hat{\sigma}^2) = \frac{m}{s\sqrt{\Delta}}\hat{\sigma} + \frac{1}{2}\left(1 - \frac{\Delta}{s^2}\right)\hat{\sigma}^2.$$

Inserting $\hat{\mu}(\hat{\sigma}^2)$ into Equation (4.16) and solving for $\hat{\sigma}^2$ completes the proof. $\diamondsuit$

Unfortunately, all the estimation methods for the models of Chesney and Taschini (2008) and Grüll and Kiesel (2009) (cf. Theorem 4.3.2 - 4.3.4) cannot be used in practice. This can be explained as follows. All the discussed estimation methods have in common that for parameter estimation one would have to compute for $a_{t_1}, \ldots, a_{t_n}$ the values of $z_{t_i} := \Phi^{-1}(a_{t_{i+1}})\sqrt{T - t_{i+1}} - \Phi^{-1}(a_{t_i})\sqrt{T - t_i}$, calculate the empirical mean and variance of $\{z_{t_i}\}$ and then equate them to the theoretical mean and variance which is a function of the model parameters μ and σ^2. A useful estimation method should ensure that the equation can be solved for every possible combination of observed mean $m \in M \subseteq \mathbb{R}$ and variance $v \in V \subseteq \mathbb{R}^+$. In other words, the set of possible mean-variance combinations $M \times V$ should span $\mathbb{R} \times \mathbb{R}^+$. However, this is not the case as the set of possible mean-variance combinations in Theorem 4.3.2 and 4.3.3 are a line and a point, respectively. In the case of Theorem 4.3.4, it is a two-dimensional set but it does not span $\mathbb{R} \times \mathbb{R}^+$. Therefore we introduce the following reduced-form model that overcomes this difficulty.

Considering the full parameter space $\mathbb{R} \times \mathbb{R}^+$ yields a so-called reduced-form model. This means, that we assume that for $a \in \mathbb{R}$, $b^2 \in \mathbb{R}^+$ the difference of Theorem 4.3.2 - 4.3.4 is distributed as follows

$$\Phi^{-1}(a_{t+\Delta})\sqrt{T - (t + \Delta)} - \Phi^{-1}(a_t)\sqrt{T - t} \sim N\left(a\Delta, b^2\Delta\right).$$

This expression can be transformed into the following SDE

$$d\left(\Phi^{-1}(a_t)\sqrt{T - t}\right) = adt + bdW_t.$$

This SDE is referred to as the SDE of the reduced-form model of Grüll and Taschini (2009) (cf. Definition 3.3.5).

4.3b Empirical analysis of reduced-form models

Introduction

Carmona et al. (2009a) introduced an option pricing model for permits using their reduced-form model. However, an analysis of the empirical performance of reduced-form models is not provided. Therefore the following subsection analyzes the empirical performance of reduced-form models compared to standard stochastic continuous-time processes such as GBM and NIG.

Description of the data

In 2005 European policy makers launched the EU ETS, the world's largest emission trading system which covers approximately 50% of the CO_2 emissions in the European Union. The EU ETS consists of three different phases. Phase I lasted until the end of 2007. Phase II started in 2008 and ends in 2012. A third phase will start in 2013. Due to bankability restrictions between phase I and II, it is necessary to treat the price series of each phase separately - see Alberola and Chevallier (2009). As the futures market is more liquid than the spot market, in what follows we perform our model calibration analysis with price series of futures contracts maturing in December 2007 and December 2012, respectively. In the first phase the price of emission permits is characterized by a very high volatility level. The significant market correction between the end of April and the beginning of May 2006 (see Figure 4.6) occurred when emission data for the year 2005 became public showing that there was an overall overestimation of offending emissions. A long-lasting futures December 2007 price decrease, characterized by a smaller volatility, started in August 2006. Such a price behaviour is typical for permit prices at the end of a compliance period. This has to do with the fact that at compliance time the permit price can only take the values zero (overallocation) or the penalty fee (permit shortage). As the reduced-form models also have this property one should expect that they excel in capturing the observed price dynamics at the end of a compliance period. In order to test this hypothesis we split up the futures December 2007 price series into two parts. We take the period of the crash as a cutting point. Prices observed during the crash (i.e. 15 trading days) are not included into our analysis. Another effect that can be observed at the end of the comliance period is that from May, 10th 2007 transaction volumes are very low and the permit price hovers below 0.30 € remaining at the same price level for several consecutive days. We consider this special effect by performing our analysis both on the full post-crash price series and on the series that is truncated on May, 10th 2007. Finally, for phase II we consider futures contracts with maturity December 2012 from January, 2nd 2008 until August, 31th 2009. The futures permit price in this period exhibits a lower volatility level hinting at a relatively more mature market. As observable in Figure 4.6, futures December 2012 prices range from 10 € to 35 €, peaking on July, 1st 2008 at 34.38 €. So, in summing, we analyze the following four data series:

1. pre-crash phase I (22 April 2005 - 24 April 2006)
2. post-crash phase I (15 May 2006 - 17 December 2007)
3. truncated post-crash phase I (15 May 2006 - 10 May 2007)
4. phase II (2 January 2008 - 31 August 2009)

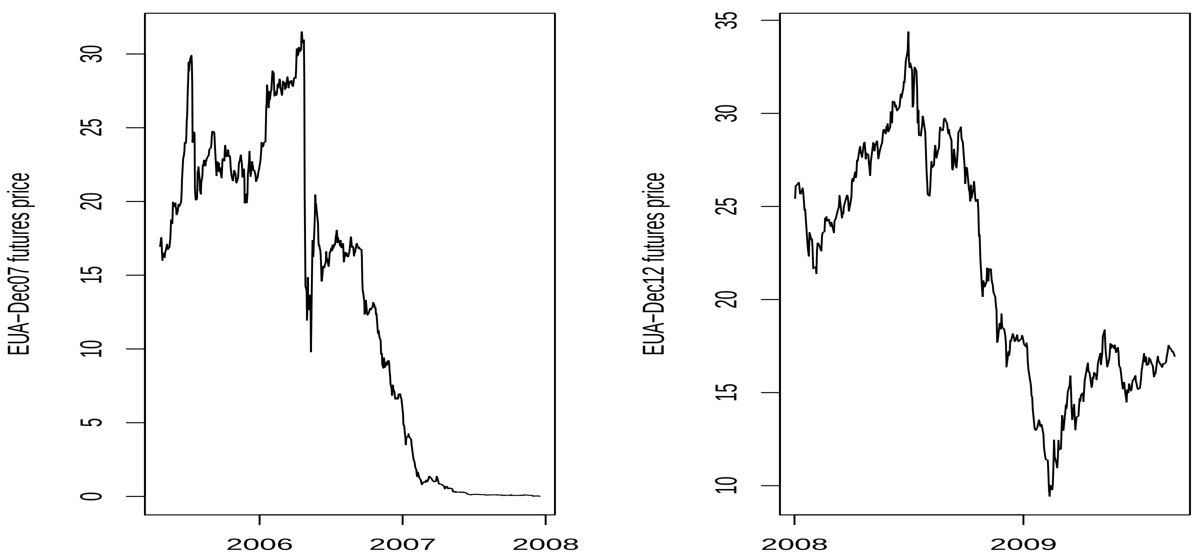

Figure 4.6: Left: EUA-Dec07 futures price (22 April 2005 - 17 December 2007), right: EUA-Dec12 futures price (2 January 2008 - 31 August 2009)

Performed analyses

Besides comparing performances of the reduced-form models of Carmona et al. (2009a) and Grüll and Taschini (2009), we calibrate other continuous-time stochastic processes and undertake an extensive model comparison. In particular, we restrict ourselves to widely known stochastic processes, such as geometric Brownian motion (GBM) and normal Inverse Gaussian (NIG). The latter is an extensively used and more complex process that overcomes some of the drawbacks of the GBM. For instance, it captures the presence of fat tails.

Because residuals of the reduced-form models and of the GBM are normally distributed, whereas residuals of the NIG process are not normally distributed, we consider two different types of analysis. We first run normality tests to all models with normally distributed residuals providing an investigation of the goodness-of-fit of reduced-form models and the GBM (cf. Table 4.2 - 4.5). Second, we assess in-sample performances of NIG, GBM and the reduced-form models of Carmona et al. (2009a) and Grüll and Taschini (2009) by comparing Q-Q-plots (cf. Figure 4.7 - 4.10) and computing the Kolmogorov-Smirnov-distance (cf. Table 4.1).
As expected, our empirical analysis shows that reduced-form models exhibit their strength at the end of a compliance period. Taking the full post-crash price series into account the reduced-form models outperform both GBM and NIG (cf. Figure 4.8 and Table 4.1 and 4.3). However, the Q-Q-plots in Figure 4.8 reveal that even reduced-form models cannot completely capture the price dynamics in this particular period. Excluding the special effect of very high volatility due to prices very close to zero and low trading volume at the very end of the first compliance period (after May, 10th 2007) we get a slightly different picture. Reduced-form models still outperform GBM but perform worse than the more complex process NIG (cf. Figure 4.9 and Table 4.1 and 4.4). At the beginning of a compliance period the price dynamics are by far captured better by NIG than the tailor-made reduced-form models. Compared to GBM, the reduced-form models perform slightly worse at the beginning of the first phase (cf. Table 4.2) and similarly at the beginning of the second phase (cf. Table 4.5). Finally, the two competing reduced-form models of Grüll and Taschini (2009) and Carmona et al. (2009a) have a similar performance whereby the model of Grüll and Taschini (2009) slightly outperforms the model of Carmona et al. (2009a) at the very end of the first compliance period (cf. Table 4.2 - 4.5). Summarizing, reduced-form models perform relatively well at the end of a compliance period compared to standard stochastic processes. However, they are clearly outperformed by complex standard stochastic processes, especially, at the beginning of the two compliance periods.

Summarized results

Using futures prices in the EU ETS with maturity December 2007 and December 2012, we calibrate reduced-form models and assess the in-sample performances of the models of Carmona et al. (2009a) and Grüll and Taschini (2009). With the aim of providing a comprehensive comparison among potentially competing models, we also calibrate and compare two quite popular continuous-time stochastic processes (GBM and NIG). In a perfect competitive equilibrium framework with no-banking options, futures permit prices are characterized by the fact that they tend to either zero or the penalty fee at the end of a compliance period. As reduced-form models capture this characteristic, we split up the permit price series in order to analyze the performance both at the beginning and at the end of a compliance period. In the current price-evolution, we observe that reduced-form models perform relatively well at the end of a compliance period compared to standard stochastic processes. However, they are clearly outperformed by complex standard stochastic processes such as NIG, especially, at the beginning of the two compliance periods. GBM and reduced-form models perform similarly at the beginning of a compliance period. However, reduced-form models describe the price dynamics at the end of the first compliance period much better than GBM. Finally, the two competing reduced-form models of Grüll and Taschini (2009) and Carmona et al. (2009a) have a similar performance whereby the model of Grüll and Taschini (2009) slightly outperforms the model of Carmona et al. (2009a) at the very end of the first compliance period.

The evaluation of the price of emission permits in the coming years will show whether, in a more mature permit market, complex standard stochastic processes such as NIG still outperform reduced-form models that take into account peculiar characteristics of permit markets.

	NIG	GBM	Carmona & Hinz	Grüll & Taschini
Phase 1 - Pre-Crash Period				
KS-Distance	**0.0321**	0.0928	0.1207	0.1179
Phase 1 - Post-Crash Period				
KS-Distance	0.1716	0.2188	0.1645	**0.1037**
Phase 1 - Post-Crash Period (truncated)				
KS-Distance	**0.0683**	0.144	0.0951	0.0994
Phase 2				
KS-Distance	**0.0257**	0.0757	0.0816	0.0785

Table 4.1: Comparison of goodness-of-fit.

Selected tables and plots

The residuals of GBM and the reduce-form models of Carmona et al. (2009a) and Grüll and Taschini (2009) are all standard normally distributed. Therefore we can apply normality tests to the log-returns in the case of GBM, to the data transformed according to the discretized version of Equation (3.85) in the case of the model of Carmona et al. (2009a) and to the data transformed according to the discretized version of Definition 3.3.5 in the case of the model of Grüll and Taschini (2009). We omit the usual footnotes concerning the significance of the normality tests as the null hypothesis that the data is normally distributed is rejected throughout at the 5% significance level. The tables show the test statistics of the performed normality tests. The most favourable test statistic for normality (i.e. the lowest) is marked bold in each row.

Normality test	GBM	Carmona & Hinz	Grüll & Taschini
Kolmogorov-Smirnov	**0.0928**	0.1207	0.1179
Anderson-Darling	**5.2260**	7.5697	7.1298
Pearson	**39.594**	67.106	67.255
Jarque-Bera	**1734.8**	3458.7	2792.4
Cramer-von-Mises	**0.8326**	1.2122	1.1363

Table 4.2: Comparison of goodness-of-fit (Pre-Crash).

Normality test	GBM	Carmona & Hinz	Grüll & Taschini
Kolmogorov-Smirnov	0.2188	0.1645	**0.1037**
Anderson-Darling	∞	13.213	**9.800**
Pearson	1048.5	689.3	**136.15**
Jarque-Bera	50059	406	**233**
Cramer-von-Mises	8.6221	2.6040	**1.7628**

Table 4.3: Comparison of goodness-of-fit (Post-Crash).

Normality test	GBM	Carmona & Hinz	Grüll & Taschini
Kolmogorov-Smirnov	0.1440	**0.0951**	0.0994
Anderson-Darling	10.581	**3.887**	4.277
Pearson	171.93	113.28	**58.53**
Jarque-Bera	387.94	82.39	**78.14**
Cramer-von-Mises	2.0310	**0.7166**	0.7889

Table 4.4: Comparison of goodness-of-fit (Post-Crash truncated).

Normality test	GBM	Carmona & Hinz	Grüll & Taschini
Kolmogorov-Smirnov	**0.0757**	0.0816	0.0785
Anderson-Darling	3.2396	3.3747	**3.0556**
Pearson	46.741	44.896	**43.377**
Jarque-Bera	**72.644**	212.838	140.951
Cramer-von-Mises	0.5395	0.5381	**0.4884**

Table 4.5: Comparison of goodness-of-fit (Second Phase).

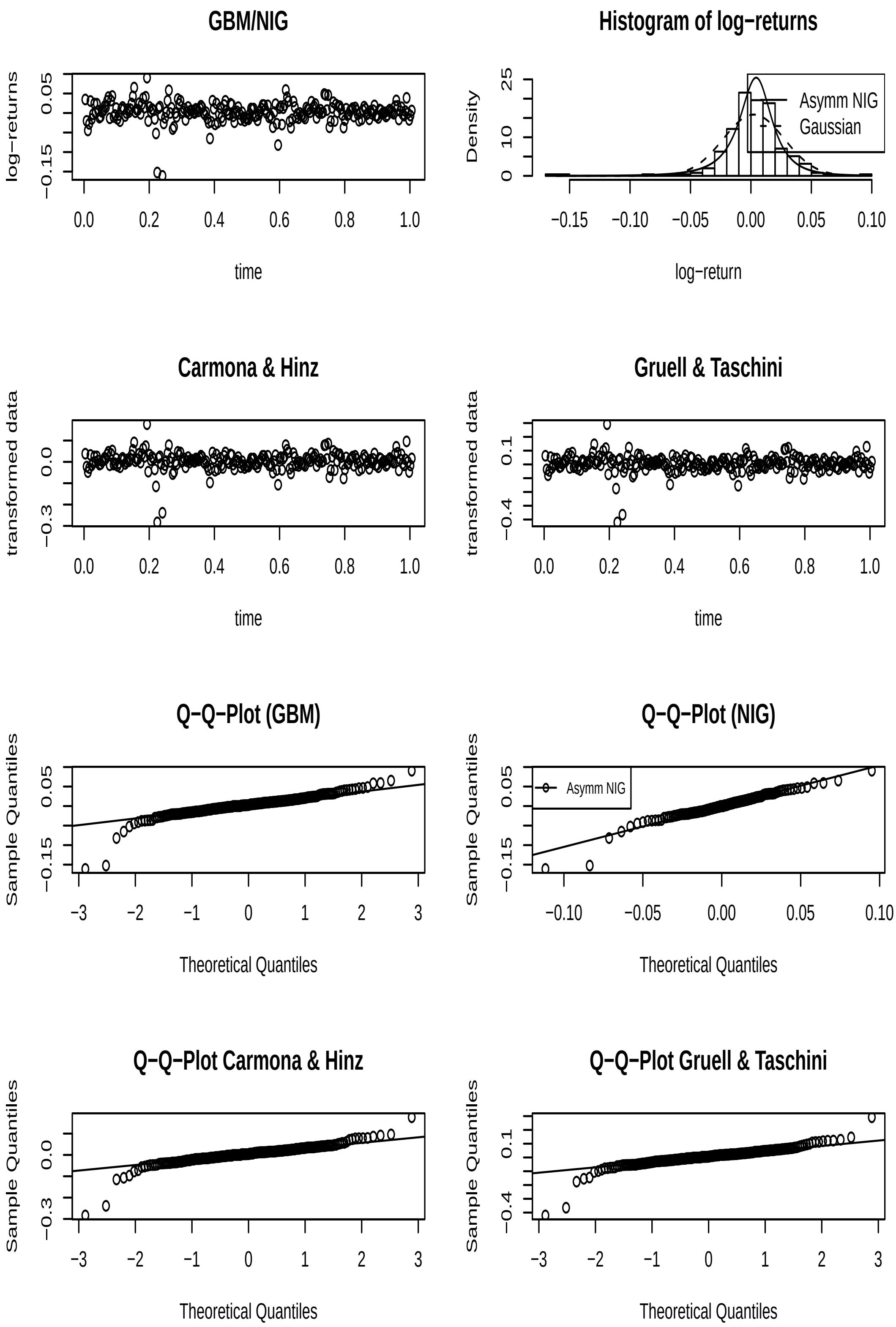

Figure 4.7: Log-returns, transformed data and Q-Q-plots of different models for pre-crash-period

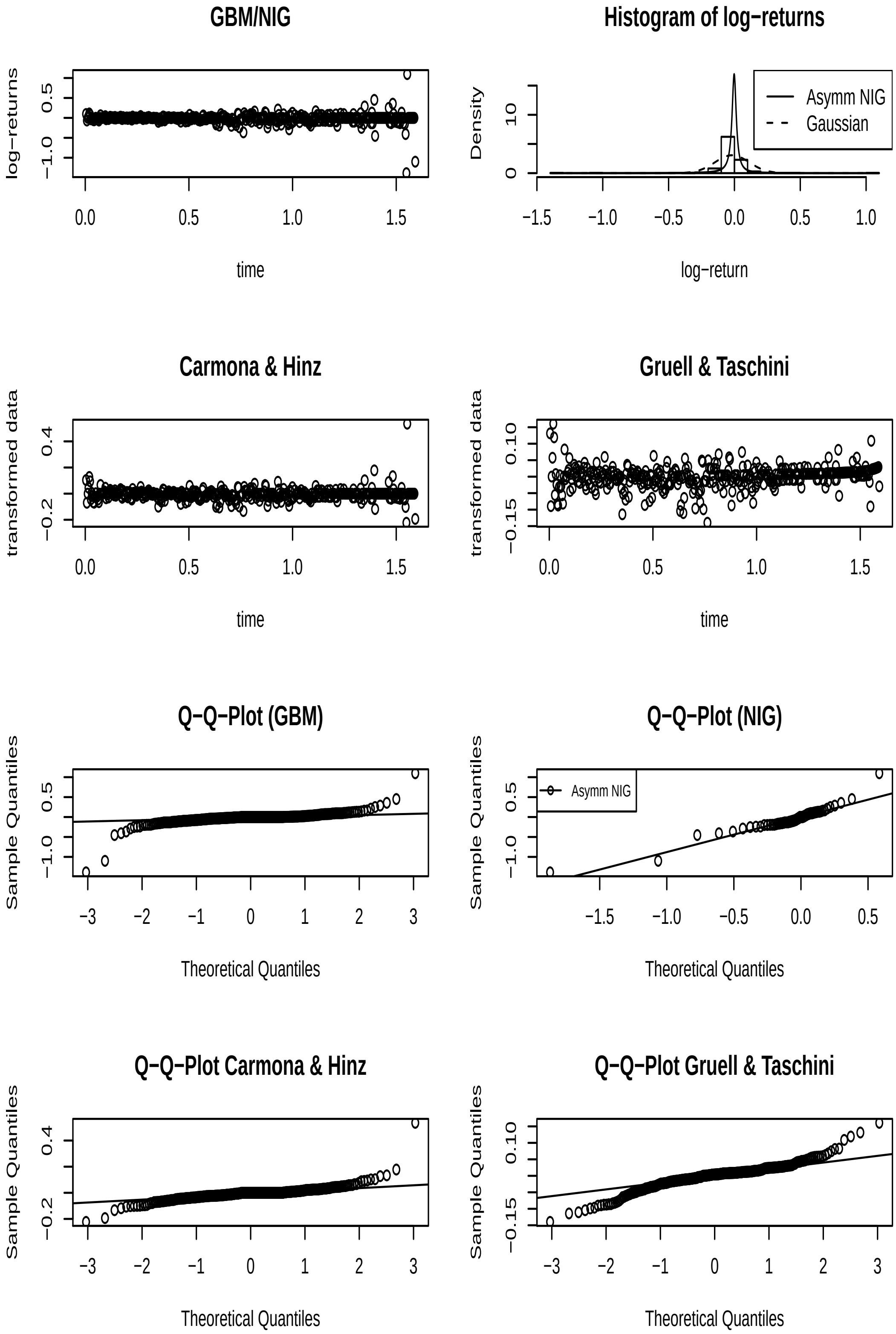

Figure 4.8: Log-returns, transformed data and Q-Q-plots of different models for post-crash-period

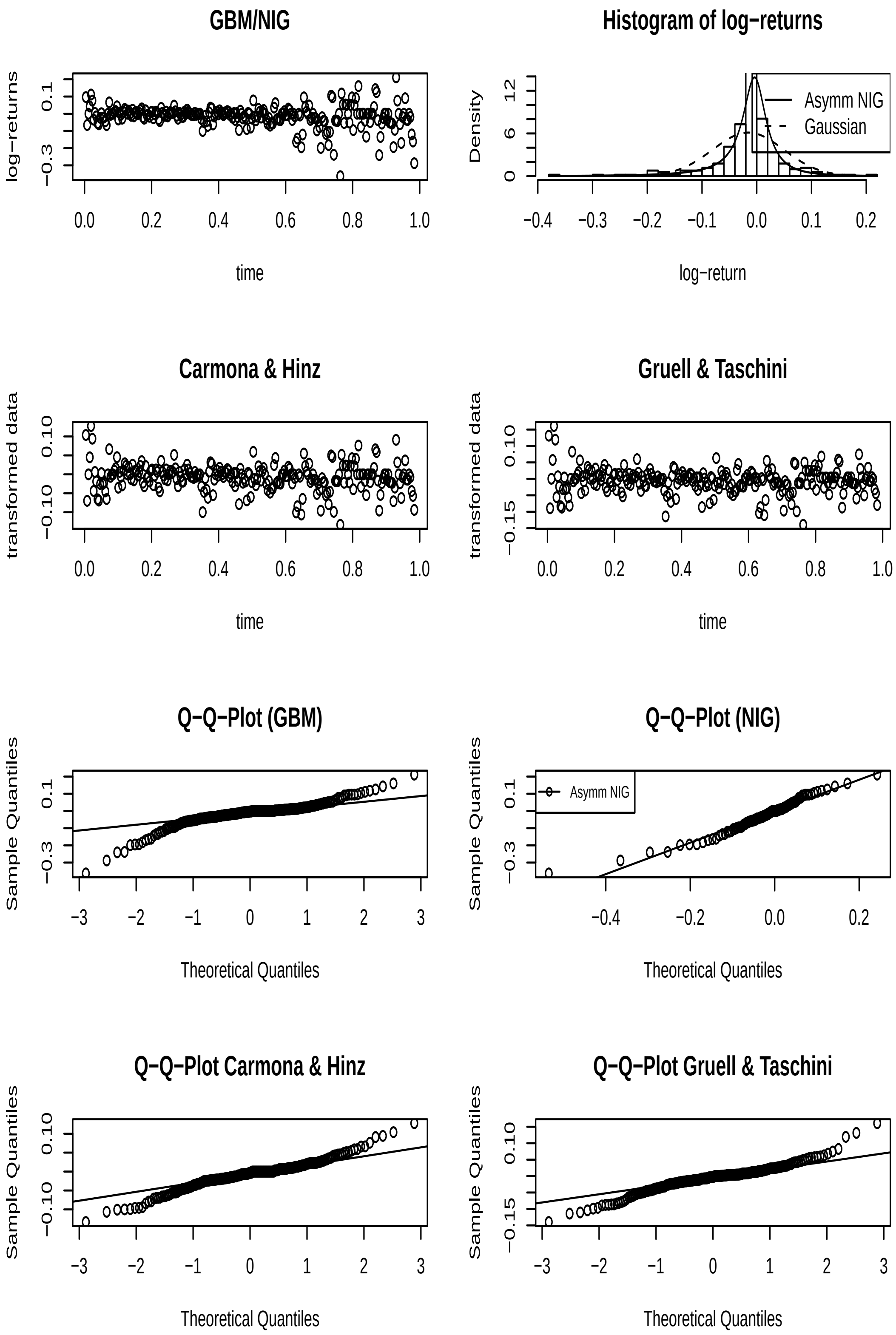

Figure 4.9: Log-returns, transformed data and Q-Q-plots of different models for post-crash-period (truncated)

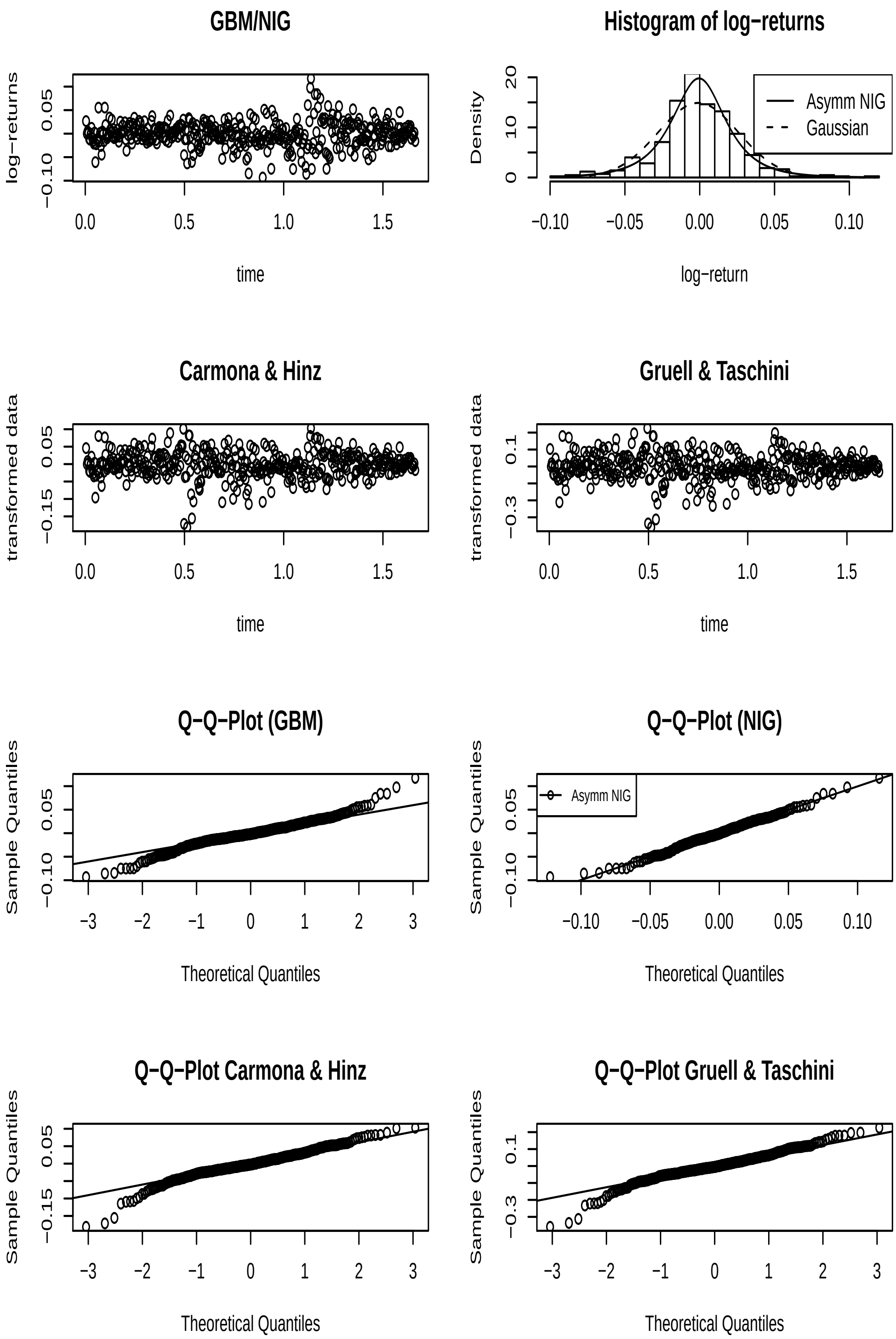

Figure 4.10: Log-returns, transformed data and Q-Q-plots of different models for second phase

Chapter 5

Permit price dynamics (Hybrid schemes)

5.1 Current and Proposed Scheme Design Mechanisms

In this section we concentrate on the most relevant scheme alternatives proposed by policy regulators to keep the permit price from rising or falling to an inordinate degree. Among suggested mechanisms, relaxing the amount of usable offsets (so-called safety-valve), introducing a subsidy to ensure a minimum price level, setting a price ceiling and price floor (so-called price collar), and creating a permit reserve to be deployed when permit prices are too high, are the most popular hybrid systems. A hybrid system is generally considered as a tailored combination of price (tax) and quantity (permit) instruments. The idea of creating a hybrid system by combining these two policy tools was first introduced by the seminal papers of Weitzman (1974) and Roberts and Spence (1976).[1] In any cap-and-trade scheme, there will be always a penalty for non-compliance. If payment of the penalty is an alternative to compliance, as in the framework of 5.1a, the penalty is effectively a price ceiling in a hybrid scheme as discussed by Jacoby and Ellerman (2004). In contrast, if payment of the penalty does not amount to compliance and the company is still obliged to comply as soon as possible, then the scheme is not directly equivalent to a conventional hybrid scheme. In the following subsections we consistently compare cap-and-trade schemes supplied with a specific mechanism (hereafter hybrid systems) to the

[1] We refer to Hepburn (2006) for a recent overview on the possible combination of price and quantity instruments.

cap-and-trade system described in Section 5.1a (hereafter ordinary system). Our analysis is performed in the one-period framework of Carmona et al. (2009b) where banking and borrowing are not allowed. Banking and borrowing options have been proposed by environmental economists with the aim of enforcing the credibility of cap-and-trade schemes and allowing a greater flexibility over time. Past literature on the analysis of how banking and borrowing mechanisms affect the price formation of emission permits is extensive. We refer to Rubin (1996) and Schennach (2000) for an analysis of the consequences of banking and borrowing on the inter-temporal trading of emission permits. In this paper we do not explicitly account for those features that would simply complicate formulae and distract the reader. However, the statements derived for the hybrid and for the ordinary systems also hold in a setting where banking is allowed.

By distinguishing the mechanisms under investigation with respect to the use of external offsets for controlling the permit price in the market, we classify the hybrid systems under study into two main groups. The first group encompasses those cap-and-trade schemes that recognize offsets as functional for compliance purposes. In particular, we study a mechanism where the amount of offsets is a function of the permit price observed in the market. The higher the permit price, the larger the amount of offsets that can be employed for compliance purposes. Conversely, the second group of scheme mechanisms relies on the ability of each policy regulator to purchase or sell an (un)limited amount of emission permits. The remainder of the hybrid systems under study belongs to this group. Neglecting possible interdependence with any offset market for the ease of exposure, we investigate these systems from Section 5.1c to Section 5.1f.

5.1a Ordinary cap-and-trade scheme

An ordinary cap-and-trade system is defined as an emissions trading scheme with the following three characteristics (cf. Section 1.1):

- At the beginning of the compliance period the regulator allocates emission allowances to the emitters for free. The number of allowances is related to the emissions of each emitter in the baseline year. This allocation method is called grandfathering.
- Allowances are freely tradeable in the compliance period.

- At the end of the compliance period emitters have to hand in one allowance per unit of emission. To enforce the cap, a penalty is levied for each unit of pollutant emitted outside the limits of a given compliance period.

Allowing for stochastic production and abatement costs, revenues from selling produced goods and emission quantities, Carmona et al. (2009b) derived the theoretical futures price of permits in the EU ETS framework, where the total pollution net of abatement reductions (the so-called aggregated cumulative emissions) in $[0, t]$ is measured by the stochastic process $q_{[0,t]}$. Let us define P as the penalty that has to be paid for each emission unit that is not covered by a permit at the compliance date T. Also, N is the total amount of permits allocated by the policy regulator to relevant companies, i.e. the cap. Both P and N are known values. We can then express a stylized version of the Carmona et al. (2009b) equilibrium price formula at time t in terms of the demand ($q_{[0,t]}$) and supply (N) of permits, and the enforcement level (P) in monetary units:

$$F(t,T) = P \cdot \mathbb{P}\left(q_{[0,T]} > N|\mathcal{F}_t\right), \tag{5.1}$$

where, after abatement reductions, $\mathbb{P}\left(q_{[0,T]} > N|\mathcal{F}_t\right)$ measures the probability of the total amount of emissions exceeding the initial amount of permits. In other words, it is the probability of the event of a shortage of permits.

In what follows, we refer to the permit price in the ordinary system by $F(t,T)$, as given in Equation (5.1). The specific variables needed to describe each hybrid system will be introduced separately in every subsection.

5.1b Safety-valve with offsets

A popular mechanism which aims to keep the price of emission permits from rising too high is the so-called safety-valve. This mechanism works by relaxing the limitations on the amount of offsets that can be used for compliance purposes. This mechanism is implemented in the Regional Greenhouse Gas Initiative (RGGI) in the United States. The RGGI is the first mandatory, market-based scheme in the United States to reduce greenhouse gas emissions. Under the RGGI ten Northeastern and Mid-Atlantic states agreed to cap and reduce their CO_2 emissions from the power sector by 10% by 2018. The RGGI allows power companies to buy offsets to meet their compliance.[2] However,

[2] A RGGI offset permit represents a project-based greenhouse gas emission reduction outside the capped electric power generation sector. The RGGI participating states limit the award of offset permits only

the use of these offsets is constrained to 3.3 percent of a power plant's total compliance obligation. The safety-valve expands this limit to 5 percent and 10 percent if given CO_2 permit price thresholds are reached in the market.

The rest of this subsection is structured as follows:
First, the framework of Carmona et al. (2009b) is extended to a situation where offsets are functional for compliance purposes (cf. Definition 5.1.1). Second, it is shown how restrictions on the use of offsets influence the permit price (cf. Theorem 5.1.2). Finally, using this extended model, we derive the theoretical pattern of the price of permits in the presence of a safety-valve with offsets, discuss the effectiveness of the safety-valve and quantify the corresponding expected enforcement costs for regulated companies.

Definition 5.1.1 (Permit price in the presence of offsets)
Assume that P is the penalty level and that N is the number of allowances handed out by the regulator.
Let $\lambda \cdot N$ be the maximum amount of offsets functional for compliance purposes.
Let the stochastic variable $c_{[0,T]}$ represent the total number of offsets that regulated companies purchase in the presence of unrestricted offset-use.
Then the stylized permit price in the presence of offsets is given by

$$\bar{F}(t,T) = P \cdot \mathbb{P}\left(q_{[0,T]} - \min\{c_{[0,T]}, \lambda N\} > N|\mathcal{F}_t\right). \tag{5.2}$$

We now derive the theoretical price bounds (lower and upper) for emission permits in the presence of restrictions on the use of offsets. This theorem is used when investigating the theoretical permit price behaviour.

Theorem 5.1.2 (Bounds for emission permit price)
Let $\lambda \in [0, \infty)$. Let $c_{[0,T]}$ be a continuous random variable on $[0, C) \subseteq [0, \infty)$. Then

(a) $\bar{F}(t,T) \in [\bar{F}_l(t,T), \bar{F}_u(t,T)]$ *where*

$$\bar{F}_l(t,T) = P \cdot \mathbb{P}\left(q_{[0,T]} > (1+\lambda)N|\mathcal{F}_t\right), \tag{5.3}$$

$$\bar{F}_u(t,T) = P \cdot \mathbb{P}\left(q_{[0,T]} > N|\mathcal{F}_t\right). \tag{5.4}$$

to five project categories. Furthermore, all offset projects must be located within one of the RGGI participating states.

(b) $\bar{F}(t,T)$ *is a non-increasing function in* λ *for* $\lambda \in [0, \frac{C}{N})$ *and constant in* λ *for* $\lambda \geq \frac{C}{N}$.

Proof :
(a) The lower and upper bound are derived by using

$$\min\{c_{[0,T]}, \lambda N\} \leq \lambda N,$$
$$\min\{c_{[0,T]}, \lambda N\} \geq 0.$$

(b) Let $c_{[0,T]}$ be a random variable on $[0, C)$. If $\lambda \geq \frac{C}{N}$, then

$$\min\{c_{[0,T]}, \lambda N\} = c_{[0,T]}.$$

Thus for $\lambda \geq \frac{C}{N}$ the permit price is equal to: $P \cdot \mathbb{P}\left(q_{[0,T]} - c_{[0,T]} > N|\mathcal{F}_t\right)$. Let $0 < \lambda < \Lambda < \frac{C}{N}$. Then we have that $\min\{c_{[0,T]}, \lambda N\} \leq \min\{c_{[0,T]}, \Lambda N\}$ almost surely which completes the proof. ♢

Assuming that the price of the offsets is solely determined by the level of emission of relevant companies, and using Equation (5.2) with time-dependent λ, the permit price is given by:

$$\bar{F}(t,T) = P \cdot \mathbb{P}\left(q_{[0,T]} - \min\left\{\lambda(t)N, c_{[0,T]}\right\} > N|\mathcal{F}_t\right), \tag{5.5}$$

where $\lambda(t)$ be an increasing step function, taking the values $0 < \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n$. As in the RGGI scheme, at each instant t the regulator allows utilities to use $\lambda(t) \cdot N$ offsets for compliance. Let $\left\{\bar{F}_1, \ldots, \bar{F}_n\right\}$ be the increasing ordered constants corresponding to permit price thresholds set by the policy regulator at the beginning of the scheme. In this framework, $t_i = \inf\left\{t, \bar{F}(t,T) = \bar{F}_i\right\}$, where $i = 1, \ldots, n$ defines the instant when the permit price $\bar{F}(t,T)$ hits the price threshold $\bar{F}_i$. Especially, we have that $\lambda(t_i) = \lambda_i$. This means that the amount of offsets that can be used for compliance at time t depends on the permit price $\bar{F}(t,T)$ observed on the market. Such a system implies that, as soon as the permit price reaches a pre-specified price barrier $\bar{F}_i$, $\lambda(\cdot)$ jumps from λ_{i-1} to λ_i.[3] This additional amount of offsets functional for compliance results in an immediate increase in the supply base of the permits and, possibly, causes a permit price drop. Looking at the price level around each instant t_i, we can observe that at time $t < t_i$ the permit price is:

$$\bar{F}(t,T) = P \cdot \mathbb{P}\left(q_{[0,T]} - \min\left\{\lambda_{i-1}N, c_{[0,T]}\right\} > N|\mathcal{F}_t\right).$$

[3]It is interesting to observe that the EU ETS implements a specific case of this mechanism. There the function $\lambda(t)$ is constant, i.e. $\lambda(t) \equiv \lambda^{EU}$, whereas in the RGGI it is an increasing step function where the step values are $\lambda_0 = 0.033, \lambda_1 = 0.05, \lambda_2 = 0.1$.

By definition, at time $t = t_i$ the permit price is equal to:

$$\bar{F}(t,T) = \bar{F}_i.$$

At time $t > t_i$, after the safety-valve has been used and the amount of offset that can be used has been increased, the permit price equals:

$$\bar{F}(t,T) = P \cdot \mathbb{P}\left(q_{[0,T]} - \min\left\{\lambda_i N, c_{[0,T]}\right\} > N|\mathcal{F}_t\right).$$

Similarly to the proof in Theorem 5.1.2, it can be shown that $\bar{F}(t,T)$ is a non-increasing function in $\lambda(t)$. However, the response of the permit price to the increase in $\lambda(\cdot)$ heavily depends on the random variable $q_{[0,T]}$. A larger amount of usable offsets, therefore, does not necessarily lead to a permit price decrease. As such, the effectiveness of this safety-valve in terms of capping a permit price increase is rather limited.

We now quantify the financial burden imposed on regulated companies by this hybrid system and compare it with an ordinary system. Let us assume that the amount λ of offsets functional for compliance purposes in the ordinary cap-and-trade system corresponds to $\lambda \equiv \lambda_0 > 0$, and $\lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n$. As we have already shown (cf. Theorem 5.1.2) that the emission permit price is a non-increasing function in λ, it is trivial to show that prices of emission permits in a hybrid system with safety-valve are lower than in an ordinary cap-and-trade system. This statement clearly implies lower expected enforcement costs for regulated companies. In other words, the overall financial burden for relevant companies in a situation of high permit prices is lowered by the presence of a mechanism that literally functions as a relief-valve for the cap-and-trade scheme.

The advantage of such a safety-valve is that it reduces expected enforcement costs for regulated companies without imposing an extra cost on the policy regulator. Furthermore, reducing the restrictions on the use of offsets is relatively easy to implement. However, as mentioned above, this mechanism does not guarantee an effective price ceiling under all circumstances. Also, and more remarkably, its success depends highly on the ability of the policy regulator to set correct price thresholds $\bar{F}_i$. This requires good skills in modeling and forecasting the supply $(c_{[0,T]})$ and demand $(q_{[0,T]})$ of permits. Finally, the fact that the amount of offsets useful for compliance purposes is a function of the (stochastic) price of emission permits, is a disadvantage for offsets project developers because it increases the overall project uncertainty.

5.1c Price Floor using a Subsidy

A severe permit price drop, followed by a price hovering above zero for more than five months during the first phase of the EU ETS, persuaded several policy makers that new cap-and-trade schemes would need additional safety-valve features. Apart from the usual presence of banking and borrowing options, therefore, policy makers have been discussing the introduction of additional mechanisms to reinforce economic incentives at the basis of market-based instruments. In particular, policy makers have been concerned about permit prices that are either too low or too high. The most obvious provision to limit such price variations is to set a price floor or ceiling. This type of mechanism will be investigated in the next section. Instead of a direct intervention on the permit price path, some economists envisage the possibility of eliminating the unfortunate consequences of extremely low permit prices by a proper combination of price (subsidy) and quantity (permit) instruments. Roberts and Spence (1976), for instance, propose to remunerate virtuous companies, i.e. companies able to massively reduce their pollution emission below their permits allocation, by means of a subsidy.

Similar to situations involving an ordinary system, a company with a permit shortage at compliance date faces a penalty P. On the contrary, when a company ends up with an excess of permits, it receives a subsidy S per unit of permit. Let $0 < S \leq P$ and let N be the initial amount of permits allocated to relevant companies. We first prove that the permit price is indeed bounded by S from below. We show that the introduction of a subsidy in fact creates a price floor equal to the subsidy. In particular, the futures permit price denoted by $\tilde{F}(t,T)$ in this hybrid system stays in the interval $[S,P]$:

$$\begin{aligned}
\tilde{F}(t,T) &= P \cdot \mathbb{P}\left(q_{[0,T]} > N \mid \mathcal{F}_t\right) + S \cdot \mathbb{P}\left(q_{[0,T]} \leq N \mid \mathcal{F}_t\right) \\
&= P \cdot \mathbb{P}\left(q_{[0,T]} > N \mid \mathcal{F}_t\right) + S \cdot \left(1 - \mathbb{P}\left(q_{[0,T]} > N \mid \mathcal{F}_t\right)\right) \\
&= S + (P-S) \cdot \mathbb{P}\left(q_{[0,T]} > N \mid \mathcal{F}_t\right) = \; S + \frac{P-S}{P} \cdot P \cdot \mathbb{P}\left(q_{[0,T]} > N \mid \mathcal{F}_t\right) \\
&= S + \frac{P-S}{P} \cdot F(t,T) = \; F(t,T) + S\left(1 - \frac{F(t,T)}{P}\right),
\end{aligned}$$

where $F(t,T) = P \cdot \mathbb{P}\left(q_{[0,T]} > N \mid \mathcal{F}_t\right)$ is the futures permit price in an ordinary system. The subsidy S, ensured by the policy regulator at the end of the compliance period, plays effectively the role of a price-floor. More interestingly, we can disentangle this hybrid scheme, emerging with an ordinary system and a European-style put option with strike price S.[4]

[4]These calculations are an alternative derivation for pricing European call and put options written on

We now quantify the financial impact on regulated companies of this hybrid system as opposed to the standard one. Let us define f_q as the probability density function of the cumulative emissions $q_{[0,T]}$ in the entire regulated period. The expected enforcement costs for relevant companies in an ordinary system are described by:

$$EEC = P\int_N^\infty (x-N)f_q(x)dx \geq 0. \tag{5.6}$$

Similarly, the expected enforcement costs for regulated companies in this hybrid system are:

$$EEC^{PF} = P\int_N^\infty (x-N)f_q(x)dx - S\int_0^N (N-x)f_q(x)dx. \tag{5.7}$$

Because $S \leq P$, a lower bound for EEC^{PF} is given by $P(\mathbb{E}[q_{[0,T]}]-N)$. Indeed,

$$EEC^{PF} \geq P\int_N^\infty (x-N)f_q(x)dx - P\int_0^N (N-x)f_q(x)dx = P(\mathbb{E}[q_{[0,T]}]-N).$$

Considering Equations (5.6) and (5.7), the total expected enforcement costs for regulated companies under this hybrid system are lower than under an ordinary system. In particular, the difference between these costs is:

$$EEC - EEC^{PF} = S\int_0^N (N-x)f_q(x)dx \ \geq 0.$$

A price floor ensured by the presence of a subsidy is relatively easy to implement and has the further advantage of lowering the expected enforcement costs for regulated companies. Furthermore, the presence of the subsidy could induce a higher stimulus in technology and abatement investments, favoring the achievement of emission reduction targets. However, the implementation of such a hybrid system might result in a significant financial burden for the environmental policy regulator. The magnitude of this burden is hardly quantifiable a priori.

emission permits with maturity corresponding to the end of the compliance period. We refer to Chesney and Taschini (2008) for the derivation of a closed-form pricing formula for European-style options on emission permits.

5.1d Price collar

The experience of the first phase of the EU ETS, and the threat of an extremely volatile price of emission permits, have persuaded several policy makers about the need for a more stable market, ideally enforceable by a strict price control mechanism. Policy makers, therefore, have been discussing the possible introduction a fixed price-range (the so-called price collar) within which the permit price can fluctuate.[5] This mechanism has long been discussed and was recently endorsed by some economists in their recommendations for a US cap-and-trade program. The rationale behind the introduction of a price collar is that the presence of a minimum (floor) and a maximum (ceiling) price of permits would lower the volatility of the price, potentially providing a higher level of price predictability. According to policy makers, such a hybrid scheme can reduce the price risk faced by innovating firms, possibly promoting higher investments in abatement technologies.

We now investigate the implications of a price collar on the trading strategies of relevant companies and on the pattern of the permit price. We use the framework of Carmona et al. (2009b) for illustration. However, the results can also be extended to more complex settings. Let P be again the penalty fee; p^{max} the price ceiling, i.e. the price at which the policy regulator sells an unlimited amount of permits; and p^{min} the price floor, i.e. the price at which the policy regulator buys an unlimited amount of permits.[6] Such a hybrid system can be broken down into a combination of an ordinary cap-and-trade system and a sum of free-of-charge American-style call and put options. In fact, when the permit price moves above a pre-specified p^{max} level, regulated companies can (have the right to) purchase at p^{max} as many permits as they need. This optionality can be quantified by summing up the values of all exercised American call options with strike price p^{max}. Similarly, when the permit price moves below a pre-specified p^{min} level, regulated companies can (have the right to) sell at p^{min} their extra permits. This optionality can be quantified by summing up the values of all exercised American put options with strike price p^{min}. However, since the amount of options on offer is unlimited, it is difficult for a policy regulator to foresee the quantity of permits needed to inject into or withdraw from the market when the permit price is respectively above p^{max} or below p^{min}. Breaking this hybrid system down into an ordinary system plus free-of-charge American options

[5] It should be noted that a price collar can be implemented also by means of a proper combination of price (tax) and quantity (permit) instruments - see Roberts and Spence (1976).

[6] When the price collar is set symmetrically around a certain permit price level $\bar{p}$, where $\bar{p} = \frac{1}{2}(p^{min} + p^{max})$, we have the so-called symmetric price collar.

maturing at compliance time, we argue that it is complex to quantify the amount of exercised American options a priori. Let N_{t-} and N_t be, respectively, the amount of outstanding permits before ($t-$) and after (t) the intervention of the policy regulator on the market for permits. Let $\alpha_t = N_t - N_{t-}$ denote the amount of permits added $(\alpha_t > 0)$ or subtracted $(\alpha_t < 0)$ to the market at time t. At each instant of time $t = 0, \dots, T$ we can identify three possible situations:

1. If the permit price is between the price collar, $F(t,T) \in (p^{min}, p^{max})$, then $\alpha_t = 0$ and there is no market intervention by the regulator on the amount of outstanding permits, i.e. $N_t = N_{t-}$.

2. If the permit price exceeds p^{max}, the policy regulator is then ready to supply an unlimited amount of additional permits. This means that regulated companies that buy permits at the price ceiling are in fact exercising American call options with a strike price p^{max}. Therefore, relying on standard arbitrage arguments, the theoretical amount of permits $\alpha_t > 0$ (corresponding to the exercised amount of American call options) that drives the market price of permits back to p^{max} is:

$$P \cdot \mathbb{P}\left(q_{[0,T]} > N_{t-} + \alpha_t | \mathcal{F}_t\right) = p^{max}$$

   The rationale behind this equality is based on a standard supply-demand mechanism: a larger supply of permits increases the downside pressure on the permit market price. However, as described below, the extra amount α_t has further potential side effects.

3. If the permit price drops below p^{min}, the policy regulator is then ready to buy an unlimited amount of permits at the price floor. This means that regulated companies that sell permits at the price floor are exercising American put options with strike price p^{min}. Similarly to the previous case, and relying on the same arbitrage arguments, the theoretical amount of permits $\alpha_t < 0$ (corresponding to the exercised amount of American put options) that drives the market price of permits up to p^{min} is:

$$P \cdot \mathbb{P}\left(q_{[0,T]} > N_{t-} + \alpha_t | \mathcal{F}_t\right) = p^{min}$$

   The supply-demand mechanism is exactly the same, but works in the opposite direction.

In an ordinary emission trading system, Equation (5.1) shows the manifest relationship between the permit price and cumulative emissions. As such, a desirable feature of the price of emission permits is that they convey most of the relevant information concerning expectations of the market about the cumulative emissions of relevant companies. This is the basic rationale behind market-based instruments: the market sets the price for scarce resources. Based on this concept, Grüll and Kiesel (2009) justify the permit price slump in 2006 in the EU ETS market that followed the publication of the verified emission data by the European Commission.[7] Intuitively, the (unknown) amount α_t has a clear impact on such a price formation mechanism. Blending in with expectations on cumulative emissions, the extra stochastic factor α_t, enhances uncertainty on the supply side and, consequently, on the permit price level. When $F(t,T) > p^{max}$, the amount of additional permits α_t that would drive the permit price back below the price ceiling in unknown prior to the compliance time. In practice regulated companies will never exercise their American call options before maturity.[8] The rationale behind such a strategy is based on the fact that companies do not physically need the permits to produce and, more importantly, they have to achieve compliance only at time T. As in the case of an American call option written on a financial underlying that pays no dividends, it is never optimal to exercise American call before maturity. Similarly, when $F(t,T) < p^{min}$, an advisable trading strategy for regulated companies is to wait until the permit price is sufficiently small, and then exercise their American put options. In such a way their put options will be more valuable or, in the financial terminology, deep in the money. So, in contrast to American call options, it makes sense to exercise American put options before maturity. Yet, the amount α_t is hardly foreseeable a priori as it depends on the option-exercising strategies of regulated companies.

Therefore, a possible side effect of α_t is intimately related to a larger uncertainty level about the net amount of permits available on the market. When $F(t,T) > p^{max}$, relevant companies with extra permits would be better off by selling their permits as soon as possible, before the regulator intervenes on the market. This action, in addition to the extra permits offered by the regulator, might result in an excessive over-supply and, consequently, in a permit price collapse. Otherwise, these companies might prefer to hold on to their permits, and wait for market price developments. This action might result in a severe decrease of permit trading volumes, possibly leading to a deadlocked market. A similar situation might occur when $F(t,T) < p^{min}$. Relevant companies would be better off holding on to their permits while the permit price stays below the price floor. Because

[7] The sudden expectation of a permit market severely in excess of permits caused an immediate price adjustment and, backed by the banking limitations in phase I, accelerated the price decrease.

[8] Because regulated companies never exercise their American call options prior to maturity, the penalty level is effectively reduced from P to p^{max}.

the price floor corresponds to American put options, as time goes by the situation cannot get worse. If the price does not recover above p^{min}, American put options will be exercised for a (guaranteed minimum) strike price equal to p^{min}. Either way, $F(t,T) > p^{max}$ or $F(t,T) < p^{min}$, the permit price will no longer reflect the real expectations of the market about the cumulative emissions of relevant companies.

The expected enforcement costs for regulated companies in a hybrid system with a price collar are lower than in an ordinary system. Intuitively, this system corresponds to an ordinary scheme with American call and put options with strike price p^{max} and p^{min}, respectively. Unfortunately, the difference in the expected enforcement costs is hardly quantifiable a priori because the regulator offers an unlimited number of additional permits at the price ceiling. However, offering an unlimited number of American call options does not result in a financial burden for the regulator. When an American call option is exercised, the regulator creates the corresponding permit (loosening its original environmental targets) and sells it for p^{max}. Conversely, the regulator faces a financial burden by offering American put options for free. When an American put option is exercised, the regulator buys back permits (leaving unaffected its original environmental targets) at a price p^{min}. As with expected enforcement costs, the determination of an a priori financial burden for the regulator is then quite complicated. As the policy regulator can at most by back the total amount of initial permits, the lower bound of the cost of this hybrid scheme can be trivially quantified as $N \cdot p^{min}$.

The price collar is a hybrid system whose objectives (setting a minimum and a maximum permit price) are always achieved. Therefore, the expected enforcement costs for regulated companies compared to an ordinary system are lower. However, this scheme has three major disadvantages. First, after the regulator market intervention, the permit price no longer reflects the real market expectations on the cumulative emissions of relevant companies. Second, the policy regulator might face severe expenses that are unquantifiable a priori or, conversely, its original environmental targets might be significantly loosened. This last consequence might be difficult to justify to public stakeholders.

5.1e Allowance Reserve

Another common mechanism proposed by economists to manage the economic (and unpopular) consequences of excessively high permit prices is to set a permit (or allowance) reserve.[9] This hybrid scheme has again been proposed by Murray et al. (2009).[10] The allowance reserve is very similar to the mechanism of the price collar. The main difference is that the maximum amount of permits available in the market equals N^{max}. In other words, the regulator sets the allowance reserve η equal to $N^{max} - N$, where $N^{max} > N$. Similar to the price collar, the allowance reserve can be broken down into an ordinary cap-and-trade system and a limited sum of free-of-charge American-style call options. In practice, when the permit price moves above a pre-specified p^{max} level, regulated companies can (have the right to) purchase permits at p^{max} up to a limited amount η. This optionality can be quantified as the value of η American call options with strike price p^{max}.

Unlike the price collar, the finite nature of the reserve η cannot guarantee the price ceiling once the reserve has been completely exploited. As opposed to the previous hybrid system, the limitation in the available extra amount of permits allows us to quantify expected enforcement costs. In particular, the difference between the expected enforcement costs of an ordinary system and the hybrid system with allowance reserve equals:

$$EEC - EEC^{AR} = (N^{max} - N)\, P^c \ \geq 0, \tag{5.8}$$

where P^c is the price of an American call option with strike price p^{max}. We can quantify an upper bound for the difference of the expected enforcement costs relying on the fact that $P^c \leq P - p^{max}$:

$$EEC - EEC^{AR} = \ \eta \cdot P^c \leq \eta \cdot (P - p^{max})\,.$$

The smaller the price ceiling, the lower the expected enforcement costs of this hybrid system. Conversely, and unsurprisingly, $EEC^{AR} = EEC$ when p^{max} tends to the penalty level P.[11]

The major disadvantage of the allowance reserve is its inability to guarantee the price ceiling once the reserve has been completely exploited. Similar to the price collar, the

[9] Here we consider situations, where the permit reserve is solely employed to control excessively high permit prices.

[10] For a comprehensive discussion of the merits of the allowance reserve, we refer to Murray et al. (2009).

[11] This corresponds to the case discussed by Jacoby and Ellerman (2004).

intervention of the regulator on the market affects the expectations of market participants regarding the cumulative emissions of relevant companies. Finally, in order to implement this scheme and partially lower the expected enforcement costs for regulated companies, the policy regulator faces new costs. Unlike the price collar, these costs are bounded. Yet, price control is possible at the expense of original environmental targets.

5.1f Plain-vanilla Options offered by the Regulator

The final mechanism under investigation concerns the offering of European- and American-style options at the inception of the compliance period for a certain price. This hybrid scheme has been proposed by Unold and Requate (2001), although they do not specify the type of options under discussion. This mechanism is closely related to the previous mechanisms (the price floor with a subsidy, the price collar and the allowance reserve). Accordingly, all these mechanisms belong to the group of hybrid systems that rely on the faculty of the policy regulator to create or withdraw permits. As described in Section 5.1c, a price floor which has been enforced using a subsidy is equivalent to an ordinary cap-and-trade system coupled with European put options. The price collar and the allowance reserve described in Sections 5.1d and 5.1e can be broken down into an ordinary system coupled with an unlimited or limited amount of American-style options. By offering standard American and/or European options at the beginning of the compliance period, a policy regulator can replicate the results enforced by a subsidy, or a price collar, or an allowance reserve. More importantly, this mechanism avoids the undesirable manipulation of expectations about the net amount of emission permits which is caused by the other hybrid systems. Clearly, as in any standard financial market, an extremely large amount of outstanding options, perhaps concentrated in the hands of few companies, might result again in undesired market price manipulation. Such an event, however unlikely, can be prevented by the policy regulator employing necessary corrective actions, such as screening options buyers.

Under the assumption that the regulator offers the options at a fair market price, the expenses borne by the regulator to implement this scheme are zero, as concluded by Unold and Requate (2001).[12] Furthermore, the permit price bounds are guaranteed for regulated

[12] More precisely, the policy regulator bears the typical risks related to writing option contracts. As a consequence, Unold and Requate (2001) raise the delicate question of whether the state or a private institution should offer these options.

companies that require this protection and are willing to pay for this optionality. Yet, the price of emission permits reflects the real expectations of market participants about the cumulative emissions of relevant companies.

5.2 Comparison of hybrid schemes

Using a stylized equilibrium permit price, we analyze five different cap-and-trade schemes characterized by specific price mechanisms. These hybrid systems are implemented by the policy regulators in order to prevent the permit price from rising too high or falling too low.

A summary of the main results is provided in Table 5.1 - 5.3. Readers preferring written text to tables are referred to 3a-3d in Section 6.1.

Scheme	Price bound	Prices can exceed bounds	Link with offsets market	Description of the mechanism
Existing cap-and-trade scheme				
Offset safety-valve	Upper	Yes	Yes	Flexible limit on the use of offsets
Proposed safety-valve mechanisms for cap-and-trade schemes				
Subsidy price floor	Lower	No	No	Subsidy
Price collar	Upper & Lower	No	No	Regulator sells unlimited amount of permits at the price ceiling and buys unlimited amount of permits at the price floor
Allowance reserve	Upper & Lower	Yes	No	Regulator sells limited amount of permits at the price ceiling and buys limited amount permits at the price floor
Regulator offers options	Upper & Lower	No (for owner of options	No	Regulator sells options at a market price

Table 5.1: Survey on the main results of the mechanisms under investigation and description of how they work in practice.

Mechanism	Advantages	Disadvantages
Offset safety valve	(a) Relatively simple to implement (b) Lower expected enforcement costs for regulated companies than in an ordinary cap-and-trade system (c) Regulator faces no financial burden	(a) Price ceiling is not guaranteed under all circumstances (b) Creates uncertainty on the projects for active emission reduction (c) Weakens the pressure for actions within the system, i.e. environmental targets are not ensured
Subsidy	(a) Relatively simple to implement (b) Reduces investment uncertainty under all circumstance (c) Stimulates reduction efforts in the system	(a) Regulator might face a significant financial burden whose size is hardly quantifiable a priori
Price collar	(a) Price collar is guaranteed under all circumstances (b) Lower expected enforcement costs for regulated companies than in an ordinary cap-and-trade system	(a) Permit prices do not reflect real expectations on the level of cumulative emissions after market intervention. The permit price volatility is not necessarily reduced (b) Regulator might face a significant financial burden when the price floor is reached (c) Regulator cannot plan the size of the financial burden and when the cash outflows will occur (d) Environmental targets are loosened when the price ceiling is reached.
Allowance reserve	(a) Compared to price collar, environmental target is only weakened up to a certain level	(a) Price bounds cannot be guaranteed under all circumstances (b) Drawbacks of price collar (see above)
Regulator offers options	(a) Regulator faces no financial burden (b) Price bounds are guaranteed for those companies willing to pay for these options (c) Environmental targets are not affected	(a) Policy regulator bears the price risk of the options written

Table 5.2: Advantages and disadvantages of the different schemes under investigation.

Mechanism		Corresponds to a combination of an ordinary cap-and-trade system and
Subsidy	-	Free of charge European-style put option with strike price equal to the price floor offered for free
Price collar	-	Free of charge American-style call option with strike price equal to the price ceiling (unlimited amount) Free of charge American-style put option with strike price equal to the price floor (unlimited amount)
Allowance reserve	-	Free of charge American-style call option with strike price equal to the price ceiling (limited amount) Free of charge American-style put option with strike price equal to the price floor (limited amount)
Options	-	European-style or American-style put and call options offered at a certain price

Table 5.3: Breaking the schemes under investigation down into an ordinary cap-and-trade system and standard financial type of options.

Chapter 6

Conclusion

6.1 Results of the thesis

1 Investigation of equilibrium models and reduced-form models

1a Marginal abatement costs and probability of permit shortage

The concept of marginal abatement costs and the concept of probability of permit shortage are closely related and the latter can be seen as an extension of the concept of marginal abatement costs.

Interpreting the permit price either as the marginal abatement costs or as the penalty multiplied by the probability of permit shortage is the result of a deterministic and a stochastic optimization problem that have a very similar structure. Regulated companies are maximizing their profits by choosing optimal strategies for emitting greenhouse gases and buying or selling permits. A by-product of solving the profit maximization problems in the different settings is that we obtain a convenient interpretation of the permit price. In a deterministic equilibrium model the permit price is equal to the marginal abatement costs (cf. Section 3.1) whereas in a stochastic equilibrium model the permit price is equal to the penalty multiplied by the probability of a permit shortage at the end of the compliance period (cf. Section 3.2). In a deterministic framework the permit price (i.e. the marginal abatement costs) does neither explicitly depend on the regulations of an emissions trading scheme such as the penalty fee and the number of allocated permits nor on the expected future emissions of the regulated companies. Permit prices in a stochastic equilibrium model capture these dependencies. At a fixed point in time the probability of permit shortage can be interpreted as the marginal abatement costs that depend on the expected cumulative emissions in the compliance period and on the total number of permits functional for compliance. This interpretation is the link between the two concepts.

1b New equilibrium model

Based on the models of Carmona et al. (2009b) and Chesney and Taschini (2008) we develop a new equilibrium model, the model of Grüll and Kiesel (2009).

Chesney and Taschini (2008) specify the process for the cumulative emissions in the framework of Carmona et al. (2009b). The emission rate of the representative agent follows a geometric Brownian motion. This implies that the total amount of pollution is described by the integral over a geometric Brownian motion. The models of Chesney and Taschini (2008) and Grüll and Kiesel (2009) differ in the way such an integral is approximated. The linear approximation approach of Chesney and Taschini (2008) has the shortcoming that the moments of the approximated cumulative emissions do not match the true ones. Grüll and Kiesel (2009) solve this problem by applying a moment matching approach.

1c Relationship of stochastic equilibrium models and reduced-form models

We show how the stochastic equilibrium model of Chesney and Taschini (2008) with time-dependent emission rate can be transformed into the reduced-form model of Carmona et al. (2009a).

The approximation of cumulative emissions in the model of Chesney and Taschini (2008) yields the reduced-form model of Carmona et al. (2009a). Choosing a more natural approximation we develop a new reduced-form model, referred to as the model of Grüll and Taschini (2009). The model of Grüll and Taschini (2009) differs slightly from the model of Carmona et al. (2009a).

2 Analysis of the price dynamics in an ordinary scheme

2a Theoretical explanation of jumpy behaviour

Permit prices are inherently prone to jumps. Apart from the scheme design, regulatory risks are also responsible for a high price volatility.

Analyzing permit price dynamics in the models of Chesney and Taschini (2008) and Grüll and Kiesel (2009) shows that permit prices are inherently prone to jumps. The extreme price slump in the EU ETS in 2006 can be explained in the equilibrium models by a relatively small change in the market's expectation for how long the remaining permits will suffice. Given a permit price time series we can compute the market's implied expectation of over-/underallocation using our equilibrium price formulae. Therefore, the price drop in 2006 of about -50% can be explained as follows: With the market being in slight implied underallocation in April 2006 (cf. Section 4.1d) rumours of a probable overallocation and the final publication of the verified emission data for 2005 on 15 May 2006 by the European Commission (overallocation of 2.5%) drove prices lower. Price jumps will occur as long as the market's ability to estimate the cumulative emission level is limited. A nearly exact estimation must have been impossible until the publication of the first verified emission data in May 2006. Even the regulator could not aggregate the data of all the countries for the first emissions report - emission data for the Czech Republic, France, the Slovak Republic and Spain was partly missing (cf. European Union (2006)). However, price jumps of the magnitude of 2006 are unlikely to occur again as the measurement of the emission data has been significantly improved.

The scheme design is not the only source of price volatility. Emissions trading schemes are surrounded by regulatory risks. Changes in the regulation or even expected or feared changes have a significant influence on prices. As pointed out in Section 2.2b and 2.2g in detail, for instance the following two regulatory risks have been responsible for permit price slumps in the EU ETS: (i) changes in the cap even during the compliance period and (ii) influence of the reduction commitments of other countries on the reduction target of the European Union.

2b Price convergence at the end of the compliance period

The concept of probability of shortage resulting from stochastic equilibrium models explains that prices in an emissions trading system without banking will always converge to zero or to the penalty at the end of the compliance period.

At the end of the compliance period there is no uncertainty about the cumulative emissions in the compliance period. Therefore, the probability of shortage only takes the values zero or one at the end of the compliance period. Multiplying this probability with the penalty yields the permit price in a stochastic equilibrium model.

2c Estimation methods for stochastic equilibrium and reduced-form models

Reduced-form models exhibit their strength at the end of a compliance period but are outperformed by complex standard-stochastic processes at the beginning of a compliance period.

We derive estimation methods for the stochastic equilibrium models of Chesney and Taschini (2008) and Grüll and Kiesel (2009) and for the reduced-form model of Grüll and Taschini (2009). The resulting estimation methods for the models of Chesney and Taschini (2008) and Grüll and Kiesel (2009) cannot be used in practice. This has to do with the fact that the obtained Stochastic Differential Equations (SDE) do not possess sufficient free parameters for model-calibration and, therefore, are not flexible enough to capture the historical permit price evolution. The reduced-form models can be used in practice. Applying the reduced-form models to the historical permit price of the EU ETS shows that the reduced-form models exhibit their strength at the end of a compliance period. However, they are clearly outperformed by complex standard stochastic processes such as NIG (Normal Inverse Gaussian) at the beginning of a compliance period. GBM (geometric Brownian motion) and reduced-form models perform similarly at the beginning of a compliance period.

3 Analysis of the permit prices in the proposed hybrid schemes

3a Breakdown of hybrid systems into an ordinary system with options
With the exception of the safety-valve, all the other hybrid systems that we have investigated can be translated into an ordinary cap-and-trade scheme combined with European-style put options (price floor with a subsidy); with an unlimited amount of American-style call and put options (price collar); with a limited amount of American-style call and put options (allowance reserve); with a limited amount of European- and American-style call and put options (standard options offered by the regulator).

3b Effectiveness of price bounds
Price bounds of emission permits can be always guaranteed in a hybrid system with a subsidy or in a system with price collar. A system where the regulator sells options to regulated companies guarantees price bounds for those companies that are willing to pay for such a protection. The two other systems under study (safety-valve with offset and allowance reserve) cannot guarantee that the permit price will be capped under all possible circumstances.

3c Permit price volatility
By breaking down hybrid systems into an ordinary system with plain-vanilla options, we assess the price collar and the allowance reserve to be unable to guarantee a reduction in the volatility of the market price of emission permits. On the contrary, they might enhance the permit price volatility. After an intervention of the regulator in the permit market took place, the price of emission permits will not reflect the real expectation of market participants regarding the cumulative emissions of regulated companies. More precisely, the unknown quantity of permits released into or withdrawn from the market alters this information.

3d Enforcement costs and environmental targets

We show that all proposed hybrid systems reduce the expected economic burden of the cap-and-trade for relevant companies. By implementing these schemes the regulator faces substantial costs (price collar), limited costs (price floor, allowance reserve), or no costs at all (safety-valve with offsets). At the same time, the original environmental targets are severely loosened (price collar), or lowered (allowance reserve and safety-valve). The hybrid scheme with standard options keeps the environmental targets under control but does not impose extra costs on the policy regulator.

6.2 Key findings of the thesis

(A) Stochastic equilibrium models and reduced-form models are very useful tools in modelling permit prices and providing theoretical explanations for permit price characteristics observed in the EU ETS (European Union Emissions Trading Scheme), the world's largest emissions trading scheme.

(B) Market participants should be aware of the following peculiarities

  (i) Permit prices in an ordinary cap-and-trade system are inherently prone to jumps

  (ii) Permit prices in an emissions trading system without banking will always converge to zero or to the penalty at the end of a compliance period

(C) Hybrid systems cannot avoid the two permit price characteristics (B-i, B-ii) without creating unwanted side-effects (cf. 3b - 3d in Section 6.1). However, a cap-and-trade system where plain-vanilla options are available (written by either the regulator or by private institutions) can replicate the intentional results of the hybrid systems under investigation and at the same time can avoid their undesirable effects. We recommend, therefore, implementing an ordinary cap-and-trade system where private institutions write options on permits. The challenge in the coming years will be the creation of properly designed option contracts on emission permits backed by sufficiently liquid option markets.

6.3 Future research

This thesis focuses on explaining the major permit price characteristics of the EU ETS (especially in Phase I between 2005 and 2007) and on analyzing the permit price dynamics in proposed hybrid schemes (safety-valve, price floor with a subsidy, price collar, allowance reserve, standard options offered by the regulator). Based on the results of the thesis my future research activity will focus on analyzing permit price dynamics of emissions trading systems where the possibility of banking and the availability of options significantly influence the permit price. Furthermore, I plan to investigate in which way the linking of different emissions trading systems influences the permit price.

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Essen, 20. Januar 2010

Georg Grüll